TOWARD RATIONAL DECISION MAKING USING THREE DIFFERENT MCDM METHODS

by

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Chapter 1 INTRODUCTION

1.1 Overview of MCDM

Decision making is a complex process which involves multiple, conflicting, and noncommensurable decision criteria. Problems encountered in public and private sectors are always given in multiple criteria format. For example, we may select an automobile that consumes less fuel and costs less than $10,000. But often we worry whether the automobile is the safest, the most prestigestious, the most comfortable, and the highest resale value after 5 years. City planners may find an urban-development project attractive since it creates lower taxes. However, they may be concerned that if the project creates a better environment, more public services, and higher job opportunity. A business executive may favor a strategy that boosts its stock price and share of market, but he may worry if this strategy is accompanied by a good company image, good labor relationship, and favorable long term growth potential. Examples like these can go on forever. Wherever they appear, all the examples share the same characteristics: multiple, conflicting, and noncommensurable decision criteria.

1.2 Purpose Of This Study

The techniques the DM uses to reach a decision are intuitive judgement and mathematic computation. To make decisions based on a human being's intuitive judgement are the most widely practiced decision process, especially in the highest management level.

However, making decisions based on intuitive judgement can be dangerous and unreliable since real world problems are usually out of any individual's comprehesion. The second technique, on the other hand, helps the DM make rational decisions by a systematic analysis of the possible courses of action. In view of the importance of the second technique, this study will be focusing on the mathematic formulation of MCDM problems and their solution methods so that the DM can utilize this study's results to make rational decisions.

1.3 MADM vs. MODM

The problems of MCDM are widely diverse. However, they are broadly classified by Hwang and Yoon [1981] among other researchers, into two categories: Multiple Attribute Decision Making (MADM), and Multiple Objective Decision Making (MODM) The following statement from Hwang and Yoon [1981] can best describe MODM and MADM:

> "... Multiple Objective Decision Making (MODM) is not associated with the problem where the alternatives are predetermined. The thrust of these models is to design the "best" alternative by considering the various interactions within the design constraints which best satisfy the DM by way of attaining some acceptable levels of a set of some quantifiable objectives. The common characteristics of MODM methods are that they possess: (1) a set of quantifiable objectives; (2) a set of well defined constraints, (3) a process of obtaining some tradeoff information, implicit or explicit, between the stated quantifiable objectives and also between stated or unstated nonquantifiable objectives. ... The distinguishing feature of the MADM is that there is usually a limited (and countably small) number of predetermined alternatives. The alternatives have associated with them a level of the achievement of the attributes (which may not necessarily be quantifiable) based on which the final

decision is to be made. The final selection of the alternative is made with the help of inter- and intra-attribute comparisons. The comparisons may involve explicit or implicit tradeoff...".

Hwang and Yoon also provide a table for the easy comparison of MADM and MODM problems.

Table 1.1 MADM vs. MODM

	MADM	MODM
Criteria (defined by)	Attributes	objectives
Objective	Implicit (ill defined)	Explicit
Attribute	Explicit	Implicit
Constraint	Inactive (Incorporated into attributes)	Active
Alternative	Finite number, discrete(prescribed)	Infinite number, Continuous (emerging as process goes)
Interaction with DM	Not much	Mostly
Usage	Selection/Evaluation	Design

To cope with the diversified MCDM problems, researchers developed solution methods for the two categories, MADM and MODM, respectively.

1.3.1 MADM Solution Methods.

The earlier works on MADM are from many different disciplines: management science, economics, psychometrics, marketing research, applied statistics, decision theory, and so on [Hwang and Masud, 1981]. Each discipline confronted multiple criteria in quite different situations. Consequently each area had developed method(s) for its own particular usage. Hwang and Yoon [1981] collected and classified 17 methods based on the information requirements from a DM. Figure 1.1 depicts that classification scheme in great detail. Each method involves its own types of underlying assumptions, information requirement, and evaluation principles. Some methods may not fully satisfy the underlying assumptions (e.g. , complete independence among attributes for simple additive weighting method), and some may have more tedious procedures (e.g. , constructing indifference curves for hierarchical tradeoffs) than others. There are also methods fitted for a specific situation (e.g. , maximin for the pessimistic decision situation, maximax for the optimistic, disjunctive method for the selection of a specialist). Even with the same preference information, each method evaluates the problem from a different point of view (e.g. , with a set of weights, weighted average outcomes for simple additive weighting, relative

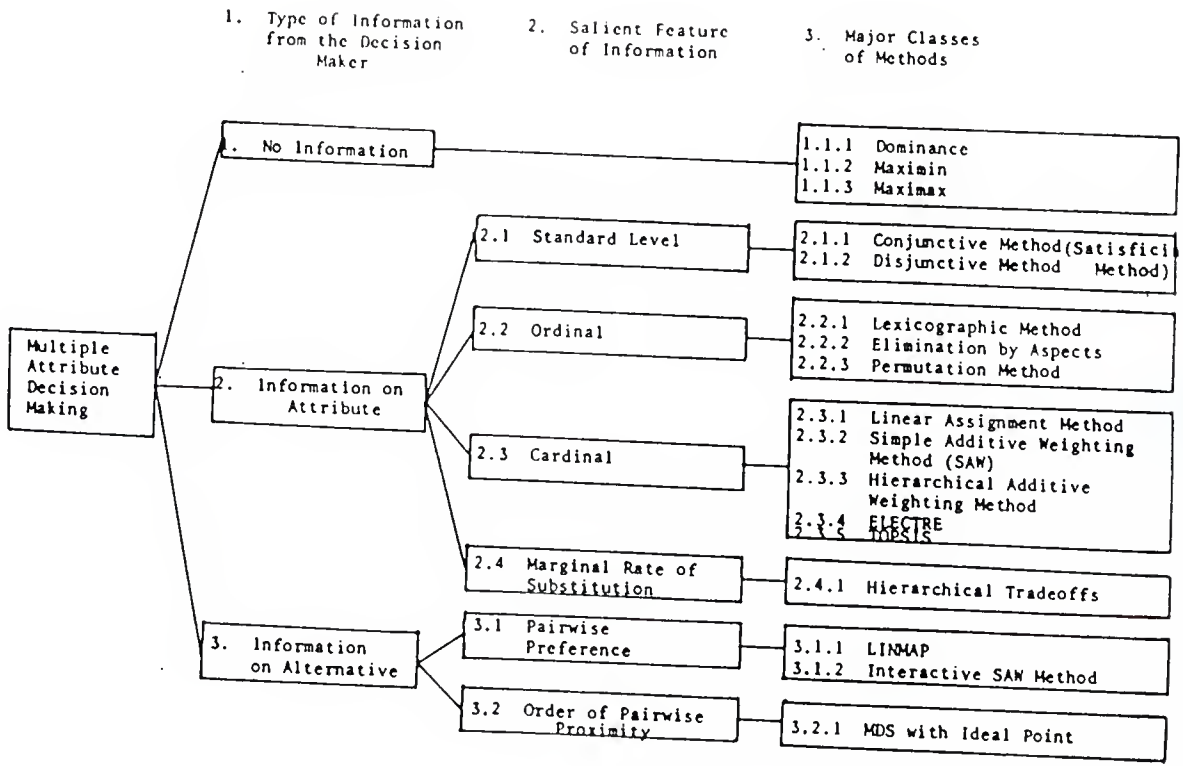


Figure 1.1 A Taxonomy of Methods for Multiple Attribute Decision Making.

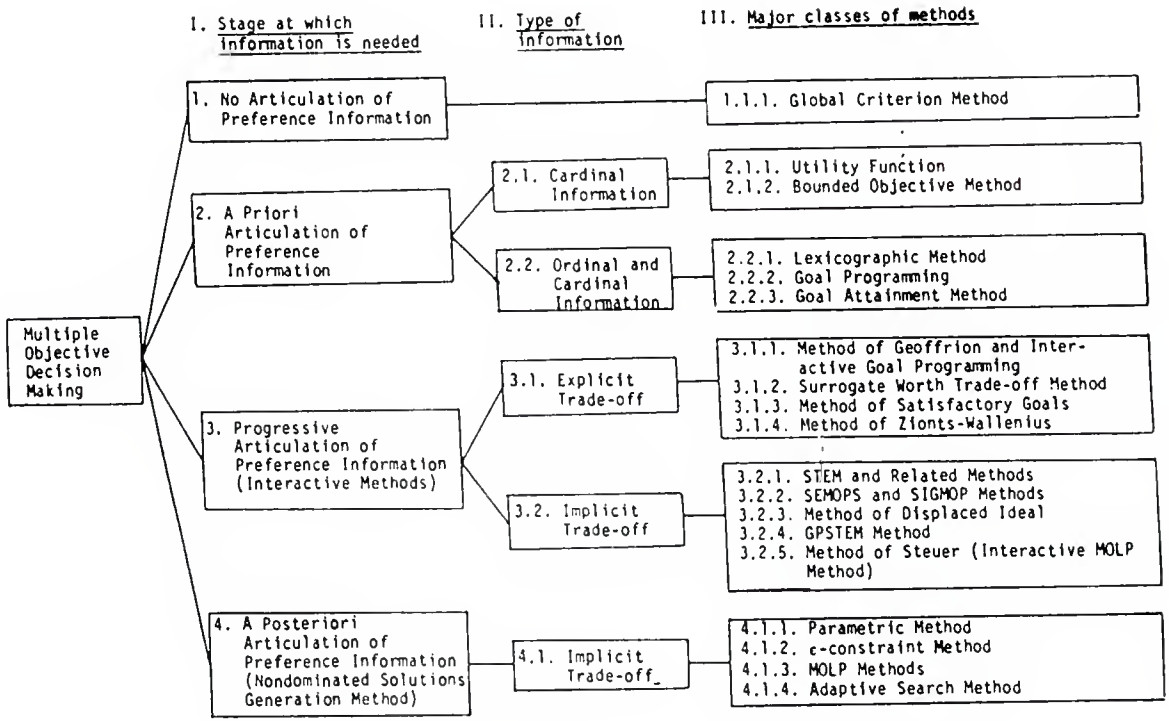
closeness to the ideal solution for TOPSIS, satisfaction to a given concordance measure for linear assignment method and ELECTRE).

It is argued by Zeleny [1982] that alternatives that are closer to the ideal are preferred to those that are farther. To be as close to the perceived ideal is the rationale of human choice. From this argument, we found that TOPSIS which was developed by Hwang and Yoon [1981] may be the the best among all the MADM methods. The reasons are that it considers not only the positive ideal point but also the negative ideal point, it is easy to calculate and to program (as a matter of fact, TOPSIS Computer software has been developed at Kansas State University), and its power to handle large problems. In this study, TOPSIS will be further improved to incorperate various normalization equations and distance measurements.

1.3.2 MODM Solution Methods.

Like MADM solution methods, the works on MODM are from various research areas. Each method has different types of assumptions, different information requirements, and are even designed for a particular type of problem. Hwang and Masud [1979] had collected and classified 19 methods. Figure 1.2 depicts that scheme of classification in detail.

The first class requires no information from the DM once the problem constraints and objectives have been defined. The analyst makes assumptions about the DM's preference and presents


I. Stage at which information is needed
II. Type of information
III. Major classes of methods
Multiple Objective Decision Making
1. No Articulation of Preference Information
1.1.1. Global Criterion Method
2. A Priori Articulation of Preference Information
2.1. Cardinal Information
2.1.1. Utility Function
2.1.2. Bounded Objective Method
2.2. Ordinal and Cardinal Information
2.2.1. Lexicographic Method
2.2.2. Goal Programming
2.2.3. Goal Attainment Method
3. Progressive Articulation of Preference Information (Interactive Methods)
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3.1.2. Surrogate Worth Trade-off Method
3.1.3. Method of Satisfactory Goals
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3.2.1. STEM and Related Methods
3.2.2. SEMOPS and SIGMOP Methods
3.2.3. Method of Displaced Ideal
3.2.4. GPSTEM Method
3.2.5. Method of Steuer (Interactive MOLP Method)
4. A Posteriori Articulation of Preference Information (Nondominated Solutions Generation Method)
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4.1.1. Parametric Method
4.1.2. ε-constraint Method
4.1.3. MOLP Methods
4.1.4. Adaptive Search Method

Figure 1.2 A Taxonomy of Methods for Multiple Objective Decision Making.

solution(s) to the DM. The second class assumes the DM has, consciously or subconciously, a set of goals to achieve and the goals set is given to the analyst before the formulation of mathematic model. The third class, also known as interactive methods, requires greater DM involvement in the solution process. The interactive takes place through a DM-analyst or DM-computer dialogue at each iteration. The tradeoff or preference information given by the DM after each iteration is used for determining a new solution. The DM, hence, actually gains insights into the problem. Finally, the fourth class does just one thing - to determine a subset of the complete set of nondominated solutions to the MODM problem. In doing so, it deals strictly with the physical constraints and makes no attempt to consider the preference of the DM. The desired outcome, however, is to narrow down the possible courses of actions and makes the selection of the preferred course of action easier.

None of the processes are superior to the others, because one method's advantages is another's shortcomings and vice versa. Therefore, two algorithms were proposed to tackle MODM problems. The first method has less DM involvement and is called TOPSIS-MODM. The second one requires great involvement of DM and is called Interactive Sequential Goal Programming II (ISGP II).

1.4 Theses Structure

Three algorithms for complex MCDM problem were attempted in this study. Chapter 1 briefly reviews the MCDM and its solution

methods. Chapter 2 describes a MADM algorithm called TOPSIS II which is a modified version of TOPSIS developed by Hwang and Yoon in 1981. Chapter 3 presents a MODM algorithm called TOPSIS-MODM which utilizes the structure of the global criteria method and the concept of TOPSIS-MADM. Chapter 4 presents an interactive MODM algorithm called ISGP II which is a refinement of Hwang and Yoon's ISGP [1982]. Finally the conclusions are given in Chapter 5.

Chapter 2 TECHNIQUE FOR ORDER PREFERENCE BY SIMILARITY TO IDEAL SOLUTION II (TOPSIS II)

2.1 Introduction

Hwang and Yoon [1981] developed the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) based upon tbe concept that the chosen alternative should have the shortest distance from the ideal soultion, A^*, and the farthest from the negative-ideal solution, A^-. That is, for a MADM problem with each attribute having a monotonically increasing (or decreasing) utility, a preferred alternative should be closest to the ideal solution -- which is composed of all the best attribute values -- and farthest from the negative-ideal solution -- which is composed of all the worst attribute values. However, it is not easy to find a preferred alternative which satisfies the foregoing situation. For example, in Fig. 2.1, an alternative A_1 has shorter distances to both the ideal solution and the negative-ideal solution than alternative A_2 does. It is then very difficult to justify the selection of A_1. To cope with this difficulty, TOPSIS measures relative closeness to the ideal solution rather than absolute distances to both ideal and negative-ideal solutions to determine the preferred alternative.

It was found later by Yoon [1986] that even with relative measurements of distance, conflicting preference alternatives may actually arise if different parameter values are used in the distance function and/ or normalization function of TOPSIS. To reconcile that conflict, modification of TOPSIS is made by introducing the credibility of a distance function, d_p.

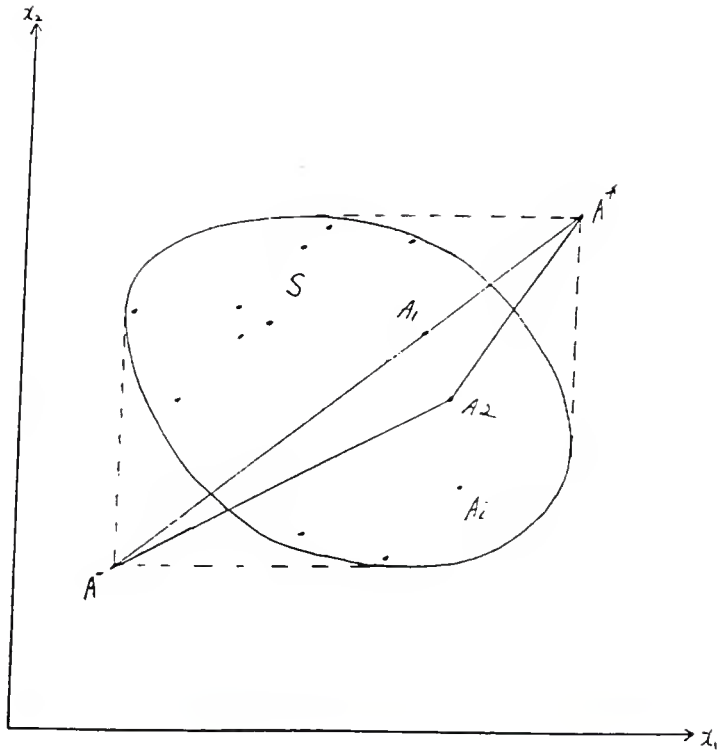

Figure 2.1 Distances to The Ideal and Negative Ideal Solutions in Two Dimension Space.

Normalization of Attribute Ratings:

For multiple attribute decision making (MADM) problems, the measurement units that each attribute possesses are different. This makes the direct comparisons among attributes impossible. Hence, the normalization of the attribute ratings is a necessary step in solving a MADM problem.

Some different normalization equations have been developed by Zeleny [1982], Hwang and Yoon [1981], and Nijkamp and Van Delft [1977]. The combination of these equations is done by Yoon [1986] using the L_p metric. If one defines the normalized scale as the ratio between individual and combined distance from the origin, the comparable scale of alternative i, A_i according to attribute j, X_j is defined as:

$$r_{ij}\ (q) = X_{ij} \ / \ \{ \sum_{i} \ X_{ij}{}^{q} \}^{1/q} \qquad (2.1)$$

where q can be any integer between 1 and oo. For q = oo, each attribute is compared by its largest attribute, i.e.:

$$r_{ij}\ (oo) = X_{ij} \ / \ \max \{ X_{ij} \} \qquad (2.2)$$

Three parameters, q = 1, 2, and oo are widely used. Expressions (2.1) and (2.2) will be used to transform the various attribute dimensions into nondimensional attributes, allowing the direct comparisons across the attributes.

Measures of Distance:

The notion "closeness" can be interpreted geometrically as the distance between two points in a n-dimension space. e.g. given points $\underline{x}^1 = (x_1{}^1, x_2{}^1, \ldots, x_n{}^1)$ and $\underline{x}^2 = (x_1{}^2, x_2{}^2, \ldots, x_n{}^2)$. the distance d between them is found to be:

$$d_p = \{ \sum_j W_j{}^p \mid x_j{}^1 - x_j{}^2 \mid^p \}^{1/p} \qquad (2.3)$$

where W_j are weights: $1 \leq p \leq oo$.

For $p = oo$, expression (2.3) is rewritten as:

$$d_{oo} = \max \{ W_j (x_j{}^1 - x_j{}^2) \} \qquad (2.4)$$

We can also see why the values p = 1, 2, and oo are especially operationally important: d_1 (the Manhattan distance or city block distance) and d_2 (the Euclidean distance) are the longest and shortest distances in the geometric sense; d_{oo} (the Tchebycheff distance) is the shortest distance in the numerical sense.

Expressions (2.5) and (2.6) are used to determine the separation measure between an alternative i to the ideal solution.

$$d_p{}^*(i) = \{ \sum_j W_j{}^p \mid r_j{}^* - r_{ij} \mid^p \}^{1/p} \qquad (2.5)$$

for $1 \leq p < oo$, and

$$d_p{}^*(i) = \{ \max w_j (r_j{}^* - r_{ij}) \}, \quad p = oo \qquad (2.6)$$

Similarly, the separation measure between alternative i to the negative ideal solution is written as:

$$d_p^-(i) = \{ \sum_j W_j^p \mid r_j^- - r_{ij} \mid^p \}^{1/p} \qquad (2.7)$$

$$d_{oo}^-(i) = \{ \max W_j \mid r_j^- - r_{ij} \mid \} \qquad (2.8)$$

Conflict arises when selection of the best alternative based on d_p^* is different from that of d_p^-. To resolve thisconflict, the relative closeness of A_i with respect to A^* is introduced and is written as:

$$D_p(i) = d_p^-(i) / \{ d_p^*(i) + d_p^-(i) \} \qquad (2.9)$$

where i = 1,2,..... m; p = 1,2, and oo .

It is clear that $0 \leq D(i) \leq 1$ where D(i) = 1 if $A_i = A^*$, and D(i) = 0 if $A_i = A^-$. An alternative A_i is relatively closer to A^* as D(i) approaches 1.

The concept of relative closeness to A^* resolves the selection problem of the reference point between A^* and A^-. TOPSIS still renders up to nine different solutions or nine sets of alternative rankings by changing the p and q parameters in both the normalization and separation computation steps. A crude resolution among conflicting rankings can be made by introducing the relative reliability rate of a distance (d_p) measurement.

Determination of Credibility of A Distance Function, d_p:

The physical property of d_p plays a key role in determining the credibility of itself. That property can be best described using indifference curves. An indifference curve is composed of many points. Any point on the same indifference curve has equal distance from the given origin. The longer the curve, the more points it can contain. That is, more points can be expressed by the same distance with a longer indifference curve. For example, in Fig. 2.2, there are three loci on a two-dimensional space with p = 1, 2, and oo. Each locus stands for an indifference curve. Notice that the locus appears farther from the origin as p increases. This shows that L_p becomes longer as p increases. Consequently, more points can be contained in the locus with a larger p or more points can be expressed by the same d_p with a larger p. We conclude that the distance function d_p becomes less specific (or less reliable) as p increases.

If we view the distance function as a measurement system, such as a grading system, temperature reading scale, and so on, the foregoing argument will be more clear. A measurement system is more credible when it has more classes. In another words, a more credible measuring system has a smaller class size. For example, the Fahrenheit scale is more credible than the centigrade scale because the former divides the temperature range between freezing and boiling into 180 degrees (classes) while the latter divides this range into only 100 degrees (classes). A temperature

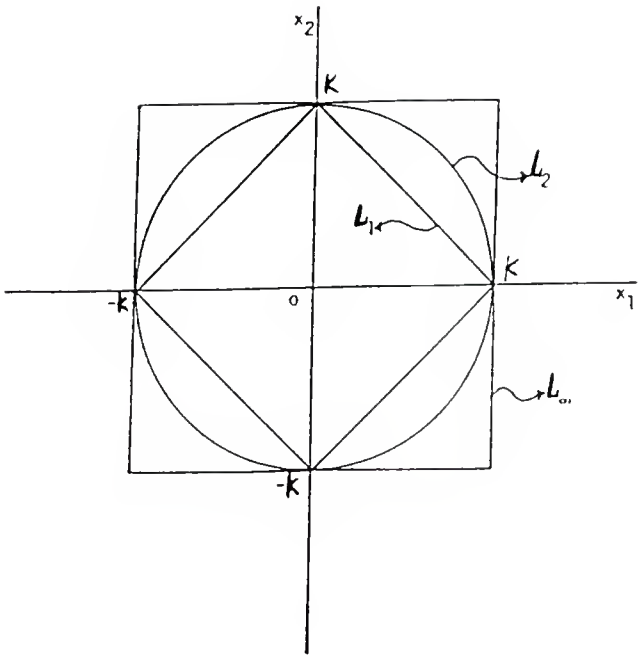

Figure 2.2 Indifference Curves with Different d_p

expressed by $20^{o}C$ can be denoted by $67^{o}F$, $68^{o}F$, or $69^{o}F$ which is more precise and reliable. Since a class contains only homogeneous elements with respect to its concern, an indifference curve is a class as far as the measurement of distance is concerned. Hence the length of an indifference curve can be viewed as a class size. Recall that the length of an indifference curve (class size) increases as p increases. Hence, we conclude the credibility of d_p decreases as parameter p increases.

Since the relative reliability of d_p is proportional to the average of d_p, $E(d_p)$, the quantification of the credibility of d_p is done by using $E(d_p)$. The procedure to obtain the values of $E(d_p)$ are as follows.

1. Generate (n * 30000) random numbers whose range is (0.0, 1.0) where n is the dimension (e.g. the number of attributes or the number of alternatives) of the problem.

1.a. Take the summation of these random numbers and divide it by 30000 to get $E(d_1)$.

1.b. Take the square of each random number and add all the results together. After the summation, take the square-root of that summation, divide it by 30000, and we get $E(d_2)$.

1.c. Group the random numbers into 30000 n-element pairs. For example, for a 2-dimensional problem, the first random number goes with the second random number, the third random number goes with the fourth random number, and so on. Compare within each group and find the largest number for that group, and add those largest numbers together. Take that summation and divide it by

30000; we get $E(d_{oo})$. Table 2.1 displays the results of the foregoing procedure.

With all the $E(d_p)$ values, the relative credibility of each d_p under the three distance system is written as:

$$C_p = E(d_p) \; / \; \{ \; E(d_1) + E(d_2) + E(d_{oo}) \; \}, \quad p = 1, 2, oo. \tag{2.10}$$

Table 2.2 displays the results for all the C_p values. (Appendix A shows a BASIC program for generating $E(d_p)$ and C_p).

Reconciling Solutions:

As expected, C_1 increases very rapidly when the number of dimensions increase. From a purely mathematic view, the use of C_1 in determining the final solution is trustworthy. However, if we consider three distance measurements simultaneously, the unified function is given as:

$$d = C_1 d_1 + C_2 d_2 + C_{oo} d_{oo}. \tag{2.11}$$

Notice that C_p values are the weights in this linear combination form and d_1 is the most heavily weighted distance measurement among all d_p.

Similarly, normalized attribute ratings get less specific when q increases. Therefore, the same set of $\underline{C}$ can also be applied in the unified normalization equation,

$$r_{ij} = C_1 \, r_{ij}(1) + C_2 \, r_{ij}(2) + C_{oo} \, r_{ij}(oo). \tag{2.12}$$

Table 2.1 Mean Values of Distance Measurements

Dimension	$E(d_1)$	$E(d_2)$	$E(d_{oo})$
2	1.0026	0.7671	0.6681
3	1.5046	0.9635	0.7520
4	1.9954	1.1199	0.7990
5	2.4913	1.2582	0.8322
6	2.9956	1.3873	0.8571
7	3.4975	1.5036	0.8756
8	4.0009	1.6115	0.8885
9	4.5043	1.7130	0.9006
10	4.9979	1.8083	0.9093
11	5.5047	1.8984	0.9180
12	6.0010	1.9820	0.9228
13	6.4949	2.0641	0.9290
14	6.9964	2.1434	0.9329
15	7.4999	2.2201	0.9374
16	8.0064	2.2954	0.9415
17	8.4895	2.3638	0.9443
18	9.0040	2.4364	0.9477
19	9.5045	2.5031	0.9497
20	9.9857	2.5662	0.9522
.	.	.	.
:	:	:	:
49	24.5066	4.0338	0.9800
50	25.0146	4.0765	0.9804

Table 2.2 The Relative Credibility of Each Distance Function

Dimension	C_1	C_2	C_{oo}
1	.3333	.3333	.3333
2	.4113	.4137	.2741
3	.4673	.2992	.2335
4	.5098	.2861	.2041
5	.5437	.2747	.1816
6	.5717	.2647	.1636
7	.5951	.2559	.1490
8	.6154	.2479	.1367
9	.6328	.2407	.1265
10	.6479	.2342	.1179
11	.6615	.2281	.1103
12	.6738	.2225	.1036
13	.6845	.2175	.0979
14	.6946	.2128	.0926
15	.7037	.2083	.0880
16	.7121	.2042	.0837
17	.7196	.2004	.0800
18	.7268	.1967	.0765
19	.7335	.1932	.0733
20	.7395	.1900	.0705
.	.	.	.
:	:	:	:
49	.8302	.1366	.0332
50	.8318	.1356	.0326

The only difference between expressions (2.11) and (2.12) is that the criteria used in determing $\underline{C}$ is different. For (2.11), $\underline{C}$ is decided by the number of attributes while $\underline{C}$ in (2.12) is determined by the number of alternatives.

2.2 Problem Formulation and Algorithm

Problem:

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccccc} X_1 & X_2 & \cdots & X_j & \cdots & X_n \end{array} \\ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1j} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2j} & \cdots & x_{2n} \\ \vdots & \vdots & & & & \vdots \\ x_{i1} & x_{i2} & \cdots & x_{ij} & \cdots & x_{in} \\ \vdots & \vdots & & & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mj} & \cdots & x_{mn} \end{bmatrix} \end{array}$$

Algorithm:

Step 1. Normalization of Problem Matrix X

$$W_j [r_{ij}(q)] = W_j [X_{ij} / \{ \sum_{i=1}^{m} X_{ij}^{q} \}^{1/q}]$$

where q can be any value between 1 and oo . i.e., $1 \leq q \leq oo$. However, the most commonly used q values are 1,2, and oo. When q = oo, each attribute is compared by only its largest attribute. That is

$$W_j [r_{ij}(oo)] = W_j [X_{ij} / \{ \sum_{i=1}^{m} X_{ij}^{oo} \}^{1/oo}]$$

can be rewritten as:

$$W_j [r_{ij}(oo)] = W_j [X_{ij} / \max \{ X_{ij} \}]$$

Step 2. Unification of Three Normalized Matrices

Different q values result indifferentnormalized matrices. The combination of the three matrices follows the formula:

$$r_{ij} = C_1\, r_{ij}(1) + C_2\, r_{ij}(2) + C_{oo}\, r_{ij}(oo)$$

where C_1, C_2 and C_{oo} are determined by the number of alternatives. See Table 3.2 for C_p values.

Step 3. Display The Unified Matrix

$$R_{ij} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccccc} X_1 & X_2 & \cdots & X_j & \cdots & X_n \end{array} \\ \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1j} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2j} & \cdots & R_{2n} \\ \vdots & & & & & \vdots \\ R_{i1} & R_{i2} & \cdots & R_{ij} & \cdots & R_{in} \\ \vdots & & & & & \vdots \\ R_{m1} & R_{m2} & \cdots & R_{mj} & \cdots & R_{mn} \end{bmatrix} \end{array}$$

Step 4. Determination of Positive-Ideal Solution and Negative-Ideal Solution

$$\underline{R}^* = \{(\max R_{ij} \mid j \in J).(\min R_{ij} \mid j \in J') \mid i = 1,2,\ldots,m\}$$

$$= \{ R_1^*, R_2^*, \ldots, R_j^*, \ldots, R_n^* \}$$

$$\underline{R}^- = \{(\min R_{ij} \mid j \in J).(\max R_{ij} \mid j \in J') \mid i = 1,2,\ldots,m\}$$

$$= \{ R_1^-, R_2^-, \ldots, R_j^-, \ldots, R_n^- \}$$

where J is the higher the better attribute, and J' is the smaller the better attribute.

Step 5. Calculation of $d_p^*(i)$ and $d_p^-(i)$

$$d_p^*(i) = \{ \sum_{j=1}^{n} W_j^p \mid R_j^* - R_{ij} \mid^p \}^{1/p}$$

$$d_p^-(i) = \{ \sum_{j=1}^{n} W_j^p \mid R_j^- - R_{ij} \mid^p \}^{1/p}$$

where p can be any integer between 1 and oo, i.e. $1 \leq p \leq oo$. However, the most commonly used p values are 1,2, and oo . When p = oo, the formulations of $d_p^*(i)$ and $d_p^-(i)$ can be rewritten as

$$d_p^*(i) = \max W_j \mid R_j^* - R_{ij} \mid$$

$$d_p^-(i) = \max W_j \mid R_j^- - R_{ij} \mid$$

Step 6. Unification of d^* and d^-

Different p values result in different distance measurements. The combination of d_p^* can be done by using:

$$d^*(i) = C_1 d_1^*(i) + C_2 d_2^*(i) + C_{oo} d_{oo}^*(i)$$

Similarly, the combination of d_p^- can be done by using:

$$d^-(i) = C_1 d_1^-(i) + C_2 d_2^-(i) + C_{oo} d_{oo}^-(i)$$

where C_1, C_2, and C_{oo} are determined by the number of attributes. See Table 2.2 for C_p values.

Step 7. Ranking The Preference Order

There are d^* and d^- values for each alternative (A_i). If d^* is used, one chooses the alternative with the smallest d^* as the

best alternative. If d^- is used, the alternative with largest d^- is thought to be the best.

However, conflict occurs when the ranking orders for d^* and d^- are different. To resolve the conflict , a unified $D = d^- / \{ d^- + d^* \}$ is introduced. The alternative with the larger D value is thought to be superior than the one with the smaller D value. Hence, a unique ranking order is obtained.

2.3 Numerical Examples

2.3.1 Selection of Fighter Aircraft

A country decided to purchase a fleet of jet fighters from the U.S. Pentagon officials offered the characteristic information of four models which may be sold to that country. The Air Force analyst team of that country agreed that six characteristics (attributes) should be considered. They are: maximum speed (X_1), ferry range (X_2), maximum payload (X_3), purchasing cost (X_4), reliability (X_5), and maneuverability (X_6). The value of the six attributes for each model (alternative) are given in Table 2.3.

Table 2.3 Decision Matrix for Aircraft Selection:

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \end{array} \\ \left[\begin{array}{cccccc} 2.0 & 1500 & 20000 & 5.5 & \text{average} & \text{very high} \\ 2.5 & 2700 & 18000 & 6.5 & \text{low} & \text{average} \\ 1.8 & 2000 & 21000 & 4.5 & \text{high} & \text{high} \\ 2.2 & 1800 & 20000 & 5.0 & \text{average} & \text{average} \end{array}\right] \end{array}$$

Before the calculation, we quantified the qualitative data using the bipolar scale developed by MacCrimmon [1968]. This scaling system is shown in figure 2.3.

For cost attributes			For benefit attributes
	0	0	
very high	1.0	1.0	very low
high	3.0	3.0	low
average	5.0	5.0	average
low	7.0	7.0	high
very low	9.0	9.0	very high
	10.0	10.0	

Fig. 2.3 Scaling System

The decision matrix thus becomes:

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \end{array} \\ \begin{bmatrix} 2.0 & 1500 & 20000 & 5.5 & 5 & 9 \\ 2.5 & 2700 & 18000 & 6.5 & 3 & 5 \\ 1.8 & 2000 & 21000 & 4.5 & 7 & 7 \\ 2.2 & 1800 & 20000 & 5.0 & 5 & 5 \end{bmatrix} \end{array}$$

Note that all attributes except X_4 are benefit criteria.

We assume the DM assigns the relative weight to each attribute as: $\underline{W}$ = (0.30, 0.20, 0.10, 0.10, 0.10, 0.20)

Step 1. Normalization of Decision Matrix

For the q = 1 case, expression (2.1) is used and the normalized weighted decision matrix is written as:

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{bmatrix} .0706 & .0375 & .0253 & .0256 & .0250 & .0692 \\ .0882 & .0675 & .0228 & .0302 & .0150 & .0384 \\ .0635 & .0500 & .0266 & .0209 & .0350 & .0538 \\ .0776 & .0450 & .0253 & .0233 & .0250 & .0384 \end{bmatrix}$$

Similarly, we obtain the second normalized weighted decision matrix for q = 2.

$$D = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \begin{bmatrix} .1402 & .0732 & .0506 & .0506 & .0481 & .1342 \\ .1752 & .1318 & .0455 & .0598 & .0298 & .0745 \\ .1261 & .0976 & .0531 & .0414 & .0674 & .1043 \\ .1542 & .0878 & .0506 & .0460 & .0481 & .0745 \end{bmatrix}$$

For q = oo , expression (2.2) is used and the matrix is written as:

$$\begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \begin{bmatrix} .2400 & .1111 & .0952 & .0846 & .0714 & .0714 \\ .3000 & .2000 & .0857 & .1000 & .0429 & .1111 \\ .2160 & .1481 & .1000 & .0692 & .1000 & .1556 \\ .2640 & .1333 & .0952 & .0769 & .0714 & .1111 \end{bmatrix}$$

Step 2. Unification of Three Normalized Weighted Decision Matrices

The combination of the three matrices is done using expression (2.12), where the relative reliability coefficients, C_q, for the four alternatives are .5098, .2861, and .2041 for q = 1, 2, and oo respectively. The unified decision matrix is written as:

$$\begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \begin{bmatrix} .1246 & .0626^{-} & .0466 & .0447 & .0410 & .1142^{*} \\ .1558^{*} & .1126^{*} & .0420^{-} & .0527^{-} & .0246^{-} & .0634^{-} \\ .1122^{-} & .0834 & .0490^{*} & .0365^{*} & .0574^{*} & .0888 \\ .1371 & .0751 & .0466 & .0406 & .0410 & .0634 \end{bmatrix}$$

Step 3. Determination of Positive-Ideal and Negative-Ideal Solutions

The positive-ideal solution is:

$$R^* = (.1558, .1126, .0490, .0365, .0574, .1142)$$

The negative-ideal solution is:

$$R^- = (.1122, .0626, .0420, .0527, .0246, .0634)$$

Step 4. Calculation Of Separation Measures

For the p = 1 case, expressions (2.5) and (2.7) are used to calculate $d_p^*(i)$ and $d_p^-(i)$ respectively, and the result is given in Table 2.4.

Table 2.4 Separation Measures When p = 1

i	$d_1^*(i)$	$d_1^-(i)$
1	.1082	.0922
2	.1068	.0936
3	.0982	.1022
4	.1299	.0705

Similarly, separation measures for the p = 2 case are obtained and are shown in Table 2.5.

Table 2.5 Separation Measures When p = 2

i	$d_2^*(i)$	$d_2^-(i)$
1	.0617	.0556
2	.0630	.0663
3	.0583	.0453
4	.0680	.0348

For the p = oo case, expressions (2.6) and (2.8) are used to calculate $d_p^*(i)$ and $d_p^-(i)$ respectively, and the result is shown in Table 2.6.

Table 2.6 Separation Measures When p = oo

i	$d_{oo}^*(i)$	$d_{oo}^-(i)$
1	.0500	.0508
2	.0508	.0500
3	.0436	.0328
4	.0508	.0249

Step 5. Unification Of Two Separation Measures, d_p^* and d_p^-

Unification of d_p^* is obtained using (2.11). Similarly, unification of d_p^- is obtained using (2.11). For six attributes, $C_1 = .5717$, $C_2 = .2647$, and $C_{oo} = .1636$ are used. The results are given in Table 2.7

Table 2.7 Unified d_p^* & d_p^- and Their Rankings

i	$d^*(i)$	ranks	$d^-(i)$	ranks
1	.0863	3	.0757	2
2	.0860	2	.0793	1
3	.0787	1	.0769	3
4	.1005	4	.0536	4

Step 6. Ranking The Preference Order

Notice the ranking order for d^* is different from the one for d^-. This conflict is resolved using expression (2.9). The values for all D's and the final ranking order is shown in Table 2.8.

Table 2.8 Final Solution

i	D(i)	Rank
1	.4673	3
2	.4797	2
3	.4942	1
4	.3478	4

The solution is:

A_3. A_2. A_1. A_4

2.3.2 Helicopter Evaluation Problem

A company currently evaluates five helicopters for its president. The quantifed data matrix is given as:

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \end{array} \\ \begin{bmatrix} 145 & 580 & 1.9 & 2625 & 3.5 & 9 & 7 & 7 \\ 190 & 500 & 2.2 & 2700 & 3.0 & 5 & 3 & 7 \\ 150 & 450 & 1.8 & 2550 & 2.5 & 9 & 9 & 5 \\ 135 & 620 & 2.5 & 2700 & 4.5 & 3 & 5 & 5 \\ 170 & 430 & 2.0 & 2600 & 4.0 & 7 & 5 & 1 \end{bmatrix} \end{array}$$

All the attributes except X_5 are benefit criteria.

We also assume the DM assigns the relative weight to each attribute as $\underline{W}$ = (1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0)

Step 1. Normalization of Decision Matrix

For the q = 1 case, expression (2.1) is used and the normalized weighted decision matrix is written as:

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{bmatrix} .1835 & .2248 & .1827 & .1992 & .2000 & .2727 & .2414 & .2800 \\ .2405 & .1938 & .2153 & .2049 & .1714 & .1515 & .1034 & .2800 \\ .1899 & .1744 & .1731 & .1935 & .1429 & .2727 & .3103 & .2000 \\ .1709 & .2403 & .2404 & .2049 & .2571 & .0909 & .1724 & .2000 \\ .2152 & .1667 & .1923 & .1973 & .2286 & .2121 & .1724 & .0400 \end{bmatrix}$$

Similarly, we obtain the second normalized weighted decision matrix for q = 2.

$$
D = \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array}
\begin{bmatrix}
.4073 & .4977 & .4056 & .4454 & .4384 & .5750 & .5092 & .5735 \\
.5337 & .4290 & .4697 & .4581 & .3757 & .3194 & .2182 & .5735 \\
.4213 & .3861 & .3843 & .4327 & .3131 & .5750 & .6547 & .4096 \\
.3792 & .5320 & .5337 & .4581 & .5636 & .1917 & .3637 & .4096 \\
.4775 & .3690 & .4270 & .4412 & .5010 & .4472 & .3637 & .0819
\end{bmatrix}
$$

For q = oo, expression (2.2) is used and the matrix is written as:

$$
D = \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array}
\begin{bmatrix}
.7632 & .9355 & .7600 & .9722 & .7778 & 1.0000 & .7778 & 1.0000 \\
1.0000 & .8065 & .8800 & 1.0000 & .6667 & .5556 & .3333 & 1.0000 \\
.7895 & .7258 & .7200 & .9444 & .5556 & 1.0000 & 1.0000 & .7143 \\
.7105 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & .3333 & .5556 & .7143 \\
.8947 & .6953 & .8000 & .9630 & .8889 & .7778 & .5556 & .1429
\end{bmatrix}
$$

Step 2. Unification of Three Normalized Weighted Decision Matrices

The combination of the three matrices is done using expression (2.12), where the relative reliability coefficients, C_q, for the five alternatives are .5437, .2747, and .1816 for q = 1, 2, and oo respectively. The unified decision matrix is written as:

$$
R = \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array}
\begin{bmatrix}
.3464 & .4242 & .3450 & .4028 & .3664 & .4825^{*} & .4078 & .4860 \\
.4540^{*} & .3656 & .4015 & .4143 & .3140 & .2681 & .1747^{-} & .4860^{*} \\
.3584 & .3291 & .3268^{-} & .3913^{-} & .2817^{*} & .4825 & .5243^{*} & .3472 \\
.3226^{-} & .4534^{*} & .4539^{*} & .4143^{*} & .4710^{-} & .1608^{-} & .2913 & .3472 \\
.4062 & .3154^{-} & .3631 & .3990 & .4187 & .3753 & .2913 & .0694^{-}
\end{bmatrix}
$$

Step 3. Determination of Positive-Ideal and Negative-Ideal Solutions

The positive-ideal alternative is:

$R^* = (\ .4540,\ .4534,\ .4539,\ .4143,\ .2817,\ .4825,\ .5243,\ .4860\ )$

The negative-ideal alternative is:

$R^- = (\ .3226,\ .3145,\ .3268,\ .3913,\ .4710,\ .1608,\ .1747,\ .0694\ )$

Step 4. Calculation of Separation Measures

Forthe p=1 case, expressions (2.5) and (2.7) are used to calculate $d_p^*(i)$ and $d_p^-(i)$ respectively, and the result is given in Table 2.9.

Table 2.9 Separation Measures When p = 1

i	$d_1^*(i)$	$d_1^-(i)$
1	.4584	1.2392
2	.7365	.9611
3	.5078	1.1888
4	1.0142	.6834
5	1.1866	.5110

Similarly, separation measures for the p = 2 case are obtained and are shown in Table 2.10.

Table 2.10 Separation Measures When p = 2

i	$d_2^*(i)$	$d_2^-(i)$
1	.2125	.5961
2	.4239	.4855
3	.2455	.5833
4	.4797	.3560
5	.5368	.2659

For the p = oo case, expressions (2.6) and (2.8) are used to calculate $d_p^*(i)$ and d_p^- respectively, and the result is shown in Table 2.11.

Table 2.11 Separation Measures When p = oo

i	$d_{oo}^*(i)$	$d_{oo}^-(i)$
1	.1165	.4166
2	.3496	.4166
3	.1388	.3496
4	.3217	.2778
5	.4166	.2145

Step 5. Unification of Two Separation Measures

Unification of d_p^* is obtained using (2.11). For eight attributes, C_1 = .6154, C_2 = .2479, and C_{oo} = .1367 are used. Similarly, unification of d_p^- is obtained using (2.11). The results are given in Table 2.12.

Table 2.12 Unified d_p^* & d_p^- and Their Rankings

i	$d^*(i)$	ranks	$d^-(i)$	ranks
1	.3536	1	.9757	1
2	.6115	3	.7756	3
3	.3956	2	.9318	2
4	.7938	4	.5516	4
5	.9282	5	.4133	5

Step 6. Ranking The Preference Order

The values for all D's and the final ranking order are shown in Table 2.13.

Table 2.13 Final Solution

i	D(i)	Rank
1	.7340	1
2	.5592	3
3	.7020	2
4	.4100	4
5	.3081	5

The solution is:

$$A_1,\ A_3,\ A_2,\ A_4,\ A_5$$

(Notice that Step 6 here is redundant, because a unique ranking order has been achieved at Step 5.)

2.3.3 Selection of Automobile

A customer wants to purchase a family car and is faced with the problem of choosing among eight automobiles. The ratings based on sixteen attributes are given in Table 2.14. The data is adopted from Fleischer [1984].

The given problem is solved by the following steps:

Step 1. Normalization of Decision Matrix

Expression (2.5) is used for the q = 1, and 2 cases. The normalized matrices are shown in Tables 2.15 and 2.16. Expression (2.6) is used for the q = oo case. The normalized matrix is given in Table 2.17.

Step 2. Unification of Three Normalized Matrices

The coefficients, C_p, (relative reliabilities) for combining matrices are determined by the number of alternatives. Hence, the unified normalized matrix is obtained using:

$$R_{ij} = .6154\ r_{ij}(1) + .2479\ r_{ij}(2) + .1367\ r_{ij}(oo).$$

Step 3. Unified Decision Matrix

The matrix calculated in Step 2 is shown in Table 2.18.

Step4. Determination of Positive-Ideal and Negative-Ideal Solutions

$\underline{R}^*$ = (.1231, .1928, .4442, .3455, .3798, .3684, .1988, .2394, .3629, .3391, .3486, .3675, .3307, .3247, .3238, .3546)

$\underline{R}^-$ = (.4421, .3295, .1110, .1234, .1799, .1973, .3515, .3222, .1561, .0484, .1307, .1135, .2480, .2680, .1835, .1773)

Table 2.14 Decision Matrix for Automobile Evaluation

Attribute	Weight	A_1 Ford Pinto	A_2 Chevrolet Nova	A_3 AMC Matador	A_4 Plymouth Fury	A_5 Cadillac El Dorado	A_6 Toyota Corona	A_7 Dodge Dart	A_8 Lincoln Continental
1.List Price ($)	1.0	2,769	3,222	3,526	4,565	9,948	3,679	3,269	9,214
2.Operating Cost ($/yr)	1.0	1,170	1,540	1,790	1,980	2,000	1,540	1,540	2,000
3.Resale value after 5 yrs	1.0	1,000	1,200	1,200	2,000	4,000	1,200	1,200	4,000
4.Maintainability (*)	1.0	5.0	5.0	7.0	2.5	6.0	7.0	5.0	6.0
5.City mileage (mpg)	1.0	18.0	12.0	10.0	9.0	11.0	19.0	12.0	10.0
6.Highway mileage (mpg)	1.0	26.0	19.0	16.0	18.0	16.0	28.0	19.0	15.0
7.Acceleration (0-60; sec)	1.0	14.7	19.1	10.8	13.0	13.7	16.2	15.3	13.5
8.Braking (60-0; ft)	1.0	180.0	165.0	155.6	141.6	178.0	170.0	165.0	190.6
9.Ride (*)	1.0	4.3	8.6	8.6	5.7	10.0	5.7	8.6	5.7
10.Handling (*)	1.0	7.0	7.0	7.0	7.0	4.0	4.0	7.0	1.0
11.Maneuverability (*)	1.0	7.0	7.0	8.0	6.0	5.0	6.0	7.0	3.0
12.Trunk space (cu.ft)	1.0	6.3	13.0	14.3	20.4	13.6	10.9	16.0	19.3
13;Front seat room (*)	1.0	7.5	7.5	10.0	7.5	10.0	7.5	7.5	10.0
14.Front seat comfort (*)	1.0	7.1	8.6	7.5	7.1	7.1	8.6	7.1	7.1
15.Rear seat room (*)	1.0	7.5	7.5	10.0	7.5	10.0	10.0	7.5	10.0
16.Rear seat comfort (*)	1.0	5.0	7.5	7.5	7.5	7.5	10.0	7.5	7.5

Notes: 1. Data from G.A. Fleischer.
2. This matrix is transposed due to space arrangement.
3. *Units from 10 point scale, from 0 (worst) to 10 (best).
4. Four attributes X_1, X_2, X_7 and X_8 are cost attributes (the larger value, the less preference).

Table 2.15 Normalized Matrix For q = 1

0.0689 0.0863 0.0633 0.1149 0.1782 0.1656 0.1264 0.1329 0.0752 0.1591 0.1429 0.0554 0.1111 0.1179 0.1071 0.0833
0.0802 0.1136 0.0759 0.1149 0.1188 0.1210 0.1642 0.1218 0.1503 0.1591 0.1429 0.1142 0.1111 0.1429 0.0107 0.1250
0.0877 0.1320 0.0759 0.1609 0.0990 0.1019 0.0929 0.1149 0.1503 0.1591 0.1633 0.1257 0.1481 0.1246 0.1429 0.1250
0.1136 0.1460 0.1266 0.0575 0.0891 0.1146 0.1118 0.1045 0.0997 0.1591 0.1224 0.1793 0.1111 0.1179 0.1071 0.1250
0.2475 0.1475 0.2532 0.1379 0.1089 0.1019 0.1178 0.1314 0.1748 0.0909 0.1020 0.1195 0.1481 0.1179 0.1429 0.1250
0.0915 0.1136 0.0759 0.1609 0.1881 0.1783 0.1393 0.1255 0.0997 0.0909 0.1224 0.0958 0.1111 0.1429 0.1429 0.1667
0.0813 0.1136 0.0759 0.1149 0.1188 0.1210 0.1316 0.1218 0.1503 0.1591 0.1429 0.1406 0.1111 0.1179 0.1071 0.1250
0.2292 0.1475 0.2532 0.1379 0.0990 0.0955 0.1161 0.1407 0.0997 0.0227 0.0612 0.1696 0.1481 0.1179 0.1429 0.1250

Table 2.16 Normalized Matrix For q = 2

0.1719 0.2408 0.1529 0.3165 0.4865 0.4569 0.3532 0.3769 0.2057 0.4217 0.3933 0.1501 0.3112 0.3333 0.3000 0.2326
0.2001 0.3169 0.1835 0.3165 0.3243 0.3339 0.4589 0.3455 0.4115 0.4217 0.3933 0.3097 0.3112 0.4038 0.3000 0.3488
0.2189 0.3684 0.1835 0.4430 0.2703 0.2812 0.2595 0.3259 0.4115 0.4217 0.4494 0.3407 0.4149 0.3521 0.4000 0.3488
0.2834 0.4075 0.3059 0.1582 0.2432 0.3163 0.3123 0.2965 0.2727 0.4217 0.3371 0.4861 0.3112 0.3333 0.3000 0.3488
0.6177 0.4116 0.6117 0.3797 0.2973 0.2812 0.3292 0.3728 0.4785 0.2410 0.2809 0.3240 0.4149 0.3333 0.4000 0.3488
0.2284 0.3169 0.1835 0.4430 0.5135 0.4921 0.3892 0.3560 0.2727 0.2410 0.3371 0.2597 0.3112 0.4038 0.4000 0.4651
0.2030 0.3169 0.1835 0.3165 0.3243 0.3339 0.3676 0.3455 0.4115 0.4217 0.3933 0.3812 0.3112 0.3333 0.3000 0.3488
0.5721 0.4116 0.6117 0.3797 0.2703 0.2636 0.3244 0.3991 0.2727 0.0602 0.1685 0.4599 0.4149 0.3333 0.4000 0.3488

Table 2.17 Normalized Matrix For q = oo

0.2783 0.5850 0.2500 0.7143 0.9474 0.9286 0.7696 0.9444 0.4300 1.0000 0.8750 0.3088 0.7500 0.8256 0.7500 0.5000
0.3239 0.7700 0.3000 0.7143 0.6316 0.6786 1.0000 0.8657 0.8600 1.0000 0.8750 0.6373 0.7500 1.0000 0.7500 0.7500
0.3544 0.8950 0.3000 1.0000 0.5263 0.5714 0.5654 0.8164 0.8600 1.0000 1.0000 0.7010 1.0000 0.8721 1.0000 0.7500
0.4589 0.9900 0.5000 0.3571 0.4737 0.6429 0.6806 0.7429 0.5700 1.0000 0.7500 1.0000 0.7500 0.8256 0.7500 0.7500
1.0000 1.0000 1.0000 0.8571 0.5789 0.5714 0.7173 0.9339 1.0000 0.5714 0.6250 0.6667 1.0000 0.8256 1.0000 0.7500
0.3698 0.7700 0.3000 1.0000 1.0000 1.0000 0.8482 0.8919 0.5700 0.5714 0.7500 0.5343 0.7500 1.0000 1.0000 1.0000
0.3286 0.7700 0.3000 0.7143 0.6316 0.6786 0.8010 0.8657 0.8600 1.0000 0.8750 0.7843 0.7500 0.8256 0.7500 0.7500
0.9262 1.0000 1.0000 0.8571 0.5263 0.5357 0.7068 1.0000 0.5700 0.1429 0.3750 0.9461 1.0000 0.8256 1.0000 0.7500

Table 2.18 Unified Normalized Matrix

0.1231 0.1928 0.1110 0.2468 0.3598 0.3421 0.2705 0.3043 0.1561 0.3391 0.3051 0.1135 0.2480 0.2680 0.2428 0.1773
0.1432 0.2537 0.1332 0.2468 0.2398 0.2500 0.3515 0.2789 0.3121 0.3391 0.3051 0.2342 0.2480 0.3247 0.1835 0.2659
0.1567 0.2949 0.1332 0.3455 0.1999 0.2105 0.1988 0.2631 0.3121 0.3391 0.3486 0.2576 0.3307 0.2832 0.3238 0.2659
0.2029 0.3262 0.2221 0.1234 0.1799 0.2368 0.2393 0.2394 0.2069 0.3391 0.2614 0.3675 0.2480 0.2680 0.2428 0.2659
0.4421 0.3295 0.4442 0.2962 0.2199 0.2105 0.2522 0.3009 0.3629 0.1938 0.2178 0.2450 0.3307 0.2680 0.3238 0.2659
0.1635 0.2537 0.1332 0.3455 0.3798 0.3684 0.2982 0.2874 0.2069 0.1938 0.2614 0.1964 0.2480 0.3247 0.3238 0.3546
0.1453 0.2537 0.1332 0.2468 0.2398 0.2500 0.2816 0.2789 0.3121 0.3391 0.3051 0.2882 0.2480 0.2680 0.2428 0.2659
0.4095 0.3295 0.4442 0.2962 0.1999 0.1973 0.2485 0.3222 0.2069 0.0484 0.1307 0.3477 0.3307 0.2680 0.3238 0.2659

Step 5. Calculation of Deviations From Ideals

When distance function parameter p is 1, the distances to the positive-ideal and the negative-ideal are summarized in Table 2.19.

Table 2.19 Separation Measurements for p = 1

i	$d_1^*(i)$	$d_1^-(i)$
1	1.2435	1.5273
2	0.9340	1.5633
3	0.7802	1.9449
4	1.0742	1.4625
5	0.3405	1.5622
6	0.7046	1.8420
7	0.9452	1.6878
8	0.6745	1.2583

Similarly, we obtain distance measurements for the p = 2 and oo cases. The results are displayed in Tables 2.20 and 2.21 respectively.

Step 6. Combination of Three Distance Measurements

The coefficients (relative reliabilities) are determined by the number of attributes. Therefore, the unified distance measurements are obtained using:

$$d = .7121\ d_1 + .2042\ d_2 + .0837\ d_{oo}$$

The results are summarized in Table 2.22.

Table 2.20 Separation Measurements for p = 2

i	$d_2^*(i)$	$d_2^-(i)$
1	0.5370	0.5607
2	0.4750	0.5328
3	0.4363	0.6096
4	0.4879	0.5203
5	0.4942	0.5254
6	0.4510	0.5554
7	0.4335	0.5513
8	0.5847	0.4947

Table 2.21 Separation Measurements for p = oo

i	$d_{oo}^*(i)$	$d_{oo}^-(i)$
1	0.3332	0.3190
2	0.3110	0.2989
3	0.3110	0.2907
4	0.2221	0.2907
5	0.3190	0.3332
6	0.3110	0.2786
7	0.3110	0.2968
8	0.2907	0.3332

Table 2.22 Unification of Distance Measurements

i	$d^*(i)$	ranks	$d^-(i)$	ranks
1	1.0230	8	1.2288	6
2	0.7881	6	1.2470	5
3	0.6707	4	1.5338	1
4	0.8832	7	1.1720	7
5	0.3701	1	1.2476	4
6	0.8199	2	1.4484	2
7	0.7876	5	1.3393	3
8	0.6240	3	1.0249	8

Step 7. Ranking The Preference Order

Notice that the ranking order for d^* is different from the one for d^-. This conflict is resolved using expression (2.9). The values for all D's and the final ranking order is given in Table 2.23.

Table 2.23 Final Solution

i	D(i)	ranks
1	0.5480	8
2	0.6127	6
3	0.6958	3
4	0.5703	7
5	0.7712	1
6	0.7003	2
7	0.6297	4
8	0.6215	5

It is clear from Table 2.23 that the preferred ranking order is:

$$A_5,\ A_6,\ A_3,\ A_7,\ A_8,\ A_2,\ A_4,\ A_1$$

2.4 Remarks

(1) TOPSIS assumes that each attribute takes either monotonically increasing or decreasing utility. This monotonicity requirement is a very reasonable assumption [Keeney and Raiffa, 1976]. A nonmonotonic utility (e.g., square utility) is a rare case, such as the optimum number of rooms in a house or the blood sugar count in a human body, where the best utility is located somewhere in the middle of an attribute range.

(2) TOPSIS considers the distance from both the positive-ideal and the negative-ideal points simultaneously by taking the relative closeness to the positive-ideal. Thus, the DM is freed from situations where whether taking the positive-ideal or the negative-ideal as reference point is uncertain. This feature is particularly appealing to the DM.

(3) The family of TOPSIS solutions may be difficult to illustrate graphically, but the behavior of choosing among the solutions can be explained by way of indifference (value) curves [Keeney and Raiffa, 1976]. A DM is assumed to give equal preference to any alternatives located on the same indifference curve.

The typical TOPSIS indifference curves for d_2 and d_{oo} are shown in Fig. 2.3 and 2.4 respectively. Any curves with $C_{i*} \geq 0.5$ are convex to the preference origin, which indicates the property of the diminishing MRS observed in most indifference curves. On the other hand, indifference curves with $C_{i*} \leq 0.5$ are concave to the preference origin. This is an unusual case of indifference curves, but it may be interpreted as a risk-prone attitude

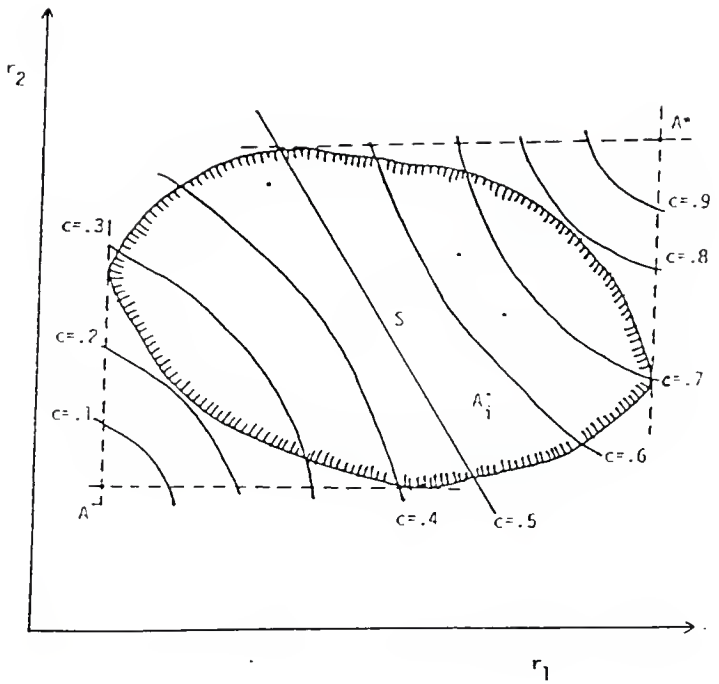

Figure 2.3 Indifference Curves in TOPSIS with d_2 Distance.

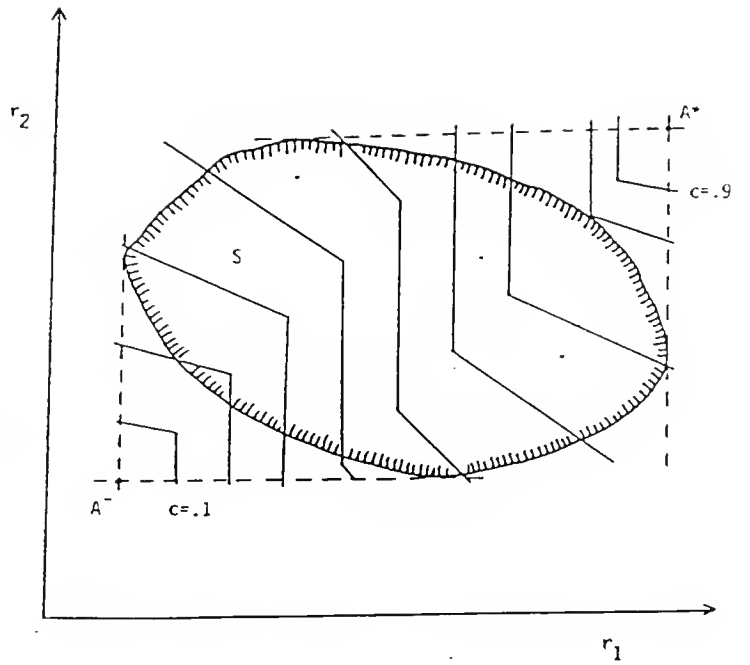

Figure 2.4 Indifference Curves in TOPSIS with d_{oo} Distance

resulting from a pessimistic situation.

(4) When we normalize attribute ratings, the parameter q affects the degree of mutual interdependency among alternatives for given set of attributes. This is especially true when q = oo, a normalized attribute rating is independent of the others except for the best rating. The lack of considering other alternative ratings leads to the lowest reliability of d_{oo}.

When we measure the distance of two alternatives, parameter p plays the role of mutual interdependency among attributes which form an alternative. When p = oo, the distance between two alternatives is independent of the other attributes except the one where the two alternatives have the largest deviation from one another. This leads to the lowest reliability of d_{oo}, too.

(5) TOPSIS solutions depend on the weight vector assigned by the DM or calculated by the analyst. Final solutions can be very different. For example, the final solution of the fighter aircraft selection problem with $\underline{w}$= (0.30, 0.20, 0.10, 0.10, 0.20) is (A3, A2, A1, A4), while the solution for $\underline{w}$=(1.0, 1.0, 1.0, 1.0, 1.0, 1.0) is (A3, A1, A4, A2).

(6) Although the TOPSIS algorithm requires tedious calculations, the understandable logic, ease of programming, and ease of computerization make it one of the best MADM methods.

Chapter 3 TOPSIS - MODM

3.1 Introduction

Hwang and Yoon [1981] developed the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) for Solving Multiple Attribute Decision Making (MADM) problems. The concept of TOPSIS was then adopted to formulate a new method for solving Multiple Objective Decision Making (MODM) problems. This new method called TOPSIS-MODM tries to find a solution that is closest to the positive ideal solution (PIS) and farthest to the negative ideal solution (NIS) by minimizing some relative distance measurements.

Relative Measure of Distance:

To measure the composite distance between two points in n-dimensions is almost impossible, because we are facing the issue of noncommensurability of individual criteria. It is necessary to make all dimensions commensurable and calculate the relative deviation instead of absolute deviation between two points. The function of the relative measure of distance between $\underline{f}^1$ and $\underline{f}^2$ can be written as:

$$d_p = \{ \sum_r W_r \left[\frac{f_r^1 - f_r^2}{f_r^* - f_r^-} \right]^p \}^{1/p} \qquad (3.1)$$

where W_r is the relative weight assigned by the DM, and p can be any integer between 1 and oo .

For p = oo , the above expression is rewritten as:

$$d_{oo} = \max \{ W_r \left[\frac{f_r^1 - f_r^2}{f_r^* - f_r^-} \right] \} \qquad (3.2)$$

The property of p is best described by Yu [1985]: "... p plays the role of the balancing factor between the "group utility sum" and the maximum of the individual regrets. If we want to emphasize the utility sum we should select a small p ; otherwise, we should select large p in order to increase the weight to the individual regrets...." Boychuk and Ovchinnikov [1973] have suggested p = 1, and Salukvadze [1972] has suggested p = 2. The most widely used p, however, are 1,2, and oo. (See Yoon [1986], Zeleny [1982] and Cohon [1978]).

The function of relative measures of distance are used later in this chapter to measure two distances: (1) the distance to the positive ideal solution, and (2) the distance to the negative ideal solution.

TOPSIS-MODM is similar to the global criteria method in the sense of finding the solution that is closest to the positive ideal solution. However, the global criteria method did not consider that a solution which is farthest to the negative ideal solution can also be a preferred solution. The composite of these two distance measurements, i.e. distance to the positive ideal and the distance to the negative ideal, are the main features of this new method.

Formulation of TOPSIS-MODM:

(i) Determine the alternative that is closest to the PIS:

$$\min F_p = \{ \sum_j W_j \left[\frac{f_j^* - f_j(\underline{x})}{f_j^* - f_j^-}\right]^p + \sum_i W_i \left[\frac{f_i(\underline{x}) - f_i^*}{f_i^- - f_i^*}\right]^p \}$$

$$\text{s.t.} \quad \underline{x} \in X \qquad (3.3)$$

(ii) Determine the alternative that is farthest to the NIS:

$$\max N_p = \{ \sum_j W_j \left[\frac{f_j(\underline{x}) - f_j^-}{f_j^* - f_j^-}\right]^p + \sum_i W_i \left[\frac{f_i^- - f_i(\underline{x})}{f_i^- - f_i^*}\right]^p \}$$

$$\text{s.t.} \quad \underline{x} \in X. \qquad (3.4)$$

The combination of equations (3.3) and (3.4) for p = 1,2....,oo cases can be generalized as:

$$\text{Min } (H_p F_p - H_n N_p) \qquad (3.5)$$

$$\text{s.t.} \quad \underline{x} \in X$$

Where H_p and H_n are the importance indexes of distances to PIS and NIS, respectively.(see next sectilon). However, when formulating (3.5) for the p = oo case, we find that it is not possible to combine (3.3) and (3.4) without transforming (3.3) to (3.6) and (3.4) to (3.7) respectively.

$$\min \; d^{*} \tag{3.6}$$

s.t. $\underline{x} \in X$

$$W_j \left[\frac{f_j^{*} - f_j(\underline{x})}{f_j^{*} - f_j^{-}} \right] \leq d^{*} \qquad (\text{All } j \in J)$$

$$W_i \left[\frac{f_i(\underline{x}) - f_i^{*}}{f_i^{-} - f_i^{*}} \right] \leq d^{*} \qquad (\text{All } i \in I)$$

and

$$\max \; d^{-} \tag{3.7}$$

s.t. $\underline{x} \in X$

$$W_j \left[\frac{f_j(\underline{x}) - f_j^{-}}{f_j^{*} - f_j^{-}} \right] \geq d^{-} \qquad (\text{All } j \in J)$$

$$W_i \left[\frac{f_i^{-} - f_i(\underline{x})}{f_i^{-} - f_i^{*}} \right] \geq d^{-} \qquad (\text{All } i \in I)$$

And merging (3.6) with (3.7), we obtain the formulation for p = oo. (see expression (3.15) for details.)

Introducing the Importance Index H_p and H_n:

As mentioned, TOPSIS-MODM tries to find a solution that is closest to the PIS and farthest from the NIS. Unfortunately, such a solution may never exist. In order to obtain a satisfactory solution, a set of importance index $\underline{H}$ (H_p, H_n) are incorporated in to the TOPSIS-MODM formulation. H_p and H_n have the following properties:

(1) H_p (H_n) is an importance index assigned by the DM to indicate the importance of choosing a solution that is closest (farthest) to the PIS (NIS).

(2) H_p and H_n are normalized weights, hence, their values will always be between 0 and 1, i.e. $0 \leq H_p \leq 1$, $0 \leq H_n \leq 1$, and the summation of these two indexes should always equal to 1, i.e. $H_p + H_n = 1$.

(3) The values of H_p and H_n may not be the same. Larger H_p implies the DM wants to use the PIS as the reference point while smaller H_p implies the DM favors the NIS to be the reference point. If H_p equals H_n, then we conclude that a compromise solution referring to both the PIS and the NIS is desired by DM.

(4) When $H_p = 1$ and $H_n = 0$, TOPSIS-MODM is the method of global criteria. In other words, the method of global criteria is a special case of TOPSIS-MODM.

(5) Although H_p and H_n values have significant effects on TOPSIS-MODM, one exception is observed. When $p = 1$ is used in the TOPSIS-MODM formulation, H_p and H_n values have no effect on the final solution, i.e. $\underline{H} = (0.75, 0.25)$ and $\underline{H} = (0.50, 0.50)$ give

identical solutions for the p = 1 case (See the first numerical example for details.)

3.2 Algorithm for TOPSIS-MODM

Problem Formulation:

Let the MODM problem be :

$$\text{Max/ Min } [\, f_1(\underline{x}),\ f_2(\underline{x}) \ldots,\ f_K(\underline{x})] \qquad (3.8)$$

$$\text{s.t. } \underline{x} \in X$$

where $X = \{\, \underline{x} \mid g_i(\underline{x})\ \{\geq, =, \leq\}\ 0,\ i = 1, 2 \ldots,\ m \,\}$

$f_j(\underline{x})$: the benefit objective for maximization, $j \in J$

$f_i(\underline{x})$: the cost objective for minimization, $i \in I$

Algorithm:

The complete TOPSIS-MODM algorithm for this problem is given below.

Step 0. Determine the Positive Ideal Solution (PIS), $\underline{f}^*$, and the Negative Ideal Solution (NIS), $\underline{f}^-$.

Step 0-1. Obtain and Construct PIS - Payoff Table

The problem is to find $\underline{x}$ so as to:

$$\max f_j(\underline{x}) \text{ , for all } j \qquad (3.9)$$

$$\text{s.t. } \underline{x} \in X$$

and / or

$$\min f_i(\underline{x}), \text{ for all } i \qquad (3.10)$$

$$\text{s.t. } \underline{x} \in X$$

The solutions of (3.9) and (3.10), f_j^* and f_i^* are known as the positive ideal solutions. Each positive ideal solution, e.g.

f_r^*, has a set of optimal decision variables, $\underline{x}^*$, substituting this set of optimal decision variables, $\underline{x}^*$, into the other objective functions f_1, f_2,....,f_{r-1}, f_{r+1}, ..., f_k, we obtain $f_{r,1}$, $f_{r,2}$,..... $f_{r,r-1}$, $f_{r,r+1}$,..... $f_{r,k}$. Then the PIS payoff table is constructed as follows:

PIS - Payoff Table

	f_1	f_2	...	f_k
f_1	f_1^*	f_{12}	...	f_{1k}
f_2	f_{21}	f_2^*	...	f_{2k}
:				
f_k	f_{k1}	f_{k2}	...	f_k^*

$$\underline{f}^* = (f_1^*, f_2^*, \ldots, f_k^*)$$

Step 0-2. Obtain and Construct NIS-Payoff Table

The problem is to find $\underline{x}$ so as to

$$\min f_j(\underline{x}), \text{ for all } j \qquad (3.11)$$

$$\text{s.t.} \quad \underline{x} \in X$$

and / or

$$\max f_i(\underline{x}), \text{ for all } i \qquad (3.12)$$

$$\text{s.t.} \quad \underline{x} \in X$$

The solution of (3.11) and (3.12), $\underline{x}^-$ and f_r^-, are known as the negative ideal solutions.

With all the $\underline{x}^-$ from (3.11) and (3.12), the NIS-Payoff table is given as:

NIS-Payoff Table

	f_1	f_2	...	f_k
f_1	f_1^-	z_{12}	...	z_{1k}
f_2	z_{21}	f_2^-	...	z_{2k}
.				
.				
.				
f_k	z_{k1}	z_{k2}	...	f_k^-

$$\underline{f}^- = (f_1^-, f_2^-, \ldots, f_k^-)$$

Note that $\underline{f}^*$ and $\underline{f}^-$ provide the upper- and lower-bounds on the objective function values and are then important reference values for the DM in all the following steps.

Step 1. Calculation phase:

The optimal solution, $\underline{x}$ and $\underline{f}$, can be obtained by solving the following problems.

(1) When p = 1. The problem is to find $\underline{x}$ so as to:

$$\min \{ H_p \left[\sum_j W_j \left[\frac{f_j^* - f_j(\underline{x})}{f_j^* - f_j^-} \right]^1 + \sum_i W_i \left[\frac{f_i(\underline{x}) - f_i^*}{f_i^- - f_i^*} \right]^1 \right] - H_n \left[\sum_j W_j \left[\frac{f_j(\underline{x}) - f_j^-}{f_j^* - f_j^-} \right]^1 + \sum_i W_i \left[\frac{f_i^- - f_i(\underline{x})}{f_i^- - f_i^*} \right]^1 \right] \}$$

$$\text{s.t.} \quad \underline{x} \in X \tag{3.13}$$

(2) When p = 2. The problem is to find $\underline{x}$ so as to:

$$\min \{ H_p \left[\sum_j W_j \left[\frac{f_j^* - f_j(\underline{x})}{f_j^* - f_j^-} \right]^2 + \sum_i W_i \left[\frac{f_i(\underline{x}) - f_i^*}{f_i^- - f_i^*} \right]^2 \right] - H_n \left[\sum_j W_j \left[\frac{f_j(\underline{x}) - f_j^-}{f_j^* - f_j^-} \right]^2 + \sum_i W_i \left[\frac{f_i^- - f_i(\underline{x})}{f_i^- - f_i^*} \right]^2 \right] \}$$

$$\text{s.t.} \quad \underline{x} \in X \tag{3.14}$$

(3) When p = oo . The problem is to find $\underline{x}$ so as to:

$$\min (d^* - d^-) \tag{3.15}$$

$$\text{s.t.} \quad \underline{x} \in X$$

$$H_p\ W_j\ \left[\frac{f_j^* - f_j(\underline{x})}{f_j^* - f_j^-}\right] \leq d^* \quad (j \in J)$$

$$H_p\ W_i\ \left[\frac{f_i(\underline{x}) - f_i^*}{f_i^- - f_i^*}\right] \leq d^* \quad (i \in I)$$

$$H_n\ W_j\ \left[\frac{f_j(\underline{x}) - f_j^-}{f_j^* - f_j^-}\right] \geq d^- \quad (j \in J)$$

$$H_n\ W_i\ \left[\frac{f_i^- - f_i(\underline{x})}{f_i^- - f_i^*}\right] \geq d^- \quad (i \in I)$$

where (a) W_i, $i = 1,2,\ldots,k$ are weights (importances) of the objectives.

(b) H_p is the DM's weight on how important it is to obtain a solution that is closest to the PIS. H_n is the DM's weight on how important it is to obtain a solution that is farthest from the NIS.

(c) $H_p + H_n = 1$

Step 2. Decision phase.

The solutions obtained from previous step are presented to the DM. The DM may select one or none of them.

3.3 Numerical examples

This section contains three numerical examples. The first one is a two-variable, two-objective linear model which is used to graphically demonstrate the power of TOPSIS-MODM. The second one is a much more complex linear problem which is more realistic than the first example. The third one is a nonlinear model to show that TOPSIS-MODM is also good for nonlinear cases.

3.3.1. Graphic Example: Hardee Company Production Scheduling

The Hardee company makes two kinds of dolls. Doll A is a high quality one and Doll B is of lower quality. The respective profits are \$0.40 and \$0.30 per doll. Each Doll A requires twice as much time as a Doll B, and if all dolls were of type B, the company could make 500 per day. The supply of material is sufficient for only 400 dolls per day (both A and B combined). The problem assumes that all the dolls for type A and type B the factory can make could be sold, and that the best customer of the company wishes to have as many as possible of type A doll. The manager realizes that two objectives: (1) the maximization of profit, and (2) the maximum production of Doll A, should be considered in scheduling the production.

This MODM problem can be formulated as follows:

$$\max f_1(\underline{x}) = 0.4x_1 + 0.3x_2$$

$$\max f_2(\underline{x}) = x_1$$

subject to

$$g_1(\underline{x}) = x_1 + x_2 \leq 400$$

$$g_2(\underline{x}) = 2\,x_1 + x_2 \leq 500$$

$$x_1,\ x_2 \geq 0$$

where x_1 and x_2 are the numbers of Doll A and that of Doll B produced respectively. And let constraints be denoted by $\underline{x} \in X$.

The steps of the TOPSIS-MODM algorithm are as follows.

Step 0. Obtain and Construct Payoff Tables

0.1 (PIS) 1) max $f_1(\underline{x}) = 0.4x_1 + 0.3x_2$

s.t. $\underline{x} \in X$

The solution is: $x_1 = 100,\ x_2 = 300;\ f_1(x^*) = 130$

Point B is the ideal solution for $f_1(\underline{x})$ (See Figs. 3.1 and 3.2).

Similarly, for the second objective:

2) max $f_2(\underline{x}) = x_1$

s.t. $\underline{x} \in X$

The solution is: $x_1 = 250,\ x_2 = 0;\ f_2(\underline{x}^*) = 250$

Point C is the ideal solution for $f_2(\underline{x})$ (See Figs. 3.1 and 3.2).

The PIS-payoff table is :

	f_1	f_2	x_1	x_2
f_1	130	100	100	300
f_2	100	250	250	0

Note that in this payoff table, row 1 gives the solution vector $\underline{x}^* = (100, 300)$ which maximizes the objective $f_1(\underline{x}^*) = 130$; $f_2(100, 300) = 100$ is the value taken on by the objective f_2 when f_1 reaches its maximum. Similarly, row 2 gives the solution vector $\underline{x}^* = (250, 0)$ which maximizes $f_2(\underline{x}^*) = 250$; $f_1(250, 0) = 100$ is the value taken on by f_1 when f_2 reaches its maximum. The point P, $(f_1^*, f_2^*) = (130, 250)$, in Figure 3.2 is an utopia solution called the PIS, which is not feasible.

0-2. (NIS) 1) $\min f_1(\underline{x}) = 0.4x_1 + 0.3x_2$

s.t. $\underline{x} \in X$

The solution is: $x_1 = 0, x_2 = 0; f_1(\underline{x}^-) = 0$

2) $\min f_2(\underline{x}) = x_1$

s.t. $\underline{x} \in X$

The solution is: $x_1 = 0, 0 \leq x_2 \leq 400; f_2(\underline{x}^-) = 0$

The NIS-payoff table is:

	f_1	f_2	x_1	x_2
f_1	0	0	0	0
f_2	120	0	0	$0 \leq x_2 \leq 400$

The NIS is point N, $(f_1^-, f_2^-) = (0, 0)$, in Figure 3.2.

Step 1. Obtain the preferred solution

A preferred solution is a nondominated solution which is a point on the segment of straight line BC. (In Figs. 3.1 and 3.2, all points on the segment of straight line BC are nondominated solutions). Assume equal weights for the two objectives and $H_p = 0.75$ and $H_n = 0.25$.

Case 1. P = 1

The problem becomes to find $\underline{x}$ so as to:

$$\min \{ (0.75) \left[\frac{130 - 0.4x_1 - 0.3x_2}{130 - 0}\right] + (0.75) \left[\frac{250 - x_1}{250 - 0}\right] - (0.25) \left[\frac{0.4x_1 + 0.3x_2 - 0}{130 - 0}\right] - (0.25) \left[\frac{x_1 - 0}{250 - 0}\right] \}$$

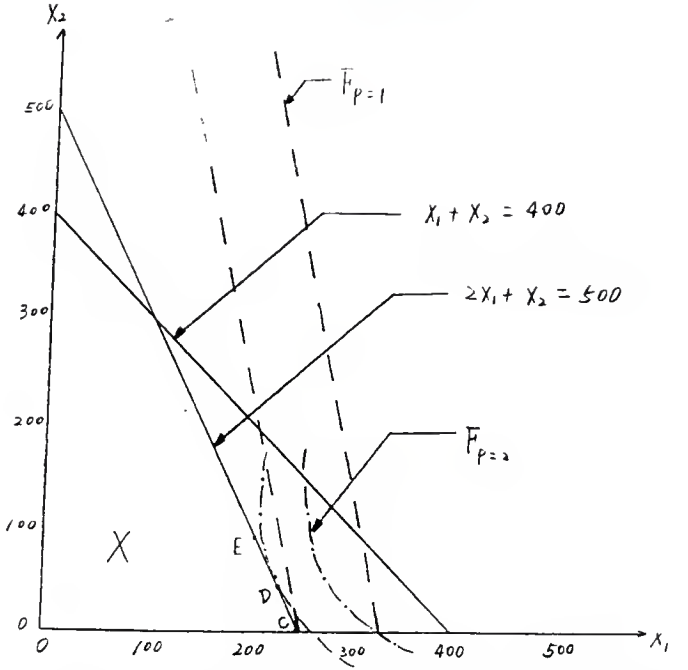

Figure 3.1 TOPSIS-MODM for Hardee Co. Production Scheduling Problem with $\underline{H}$ = (0.75, 0.25), $\underline{w}$ = (0.50, 0.50)

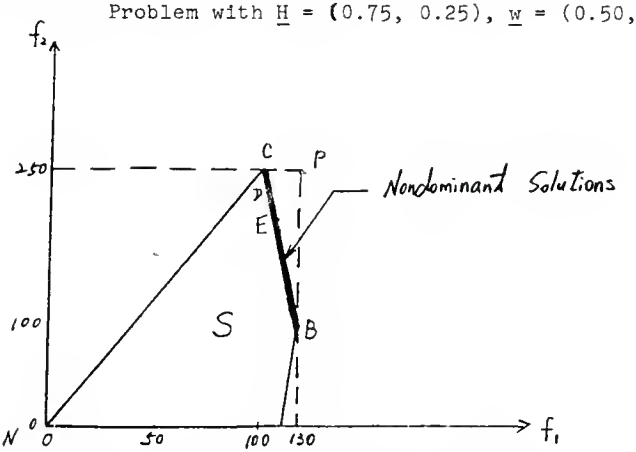

Figure 3.2 Objective Function Representation of Feasible Area of Hardee Co., Production Scheduling Problem

s.t. $\underline{x} \in X$

This is a LP problem. The solution is :

$$\underline{x} = (250, 0)$$

$$\underline{f} = (100, 250)$$

which is point C in Figs. 3.1 and 3.2.

Case 2. P = 2

The problem becomes to find $\underline{x}$ so as to:

$$\min \{ (0.75) \left[\frac{130 - 0.4x_1 - 0.3x_2}{130 - 0}\right]^2$$

$$+ (0.75) \left[\frac{250 - x_1}{250 - 0}\right]^2$$

$$- (0.25) \left[\frac{0.4x_1 + 0.3x_2 - 0}{130 - 0}\right]^2$$

$$- (0.25) \left[\frac{x_1 - 0}{250 - 0}\right]^2 \}$$

s.t. $\underline{x} \in X$

This is a nonlinear programming problem. We use Sequential Unconstrainted Minimization Technique (SUMT) to solve it. The solution is :

$$\underline{x} = (232.21, 35.61)$$

$$\underline{f} = (103.56, 232.21)$$

which is point D in Figs. 3.1 and 3.2.

Case 3. P = oo

The problem is to find $\underline{x}$ so as to:

$$\min \{ d^* - d^- \}$$

$$\text{s.t.} \quad \underline{x} \in X$$

$$(0.75) \left[\frac{130 - 0.4x_1 - 0.3x_2}{130 - 0} \right] \leq d^*$$

$$(0.75) \left[\frac{250 - x_1}{250 - 0} \right] \leq d^*$$

$$(0.25) \left[\frac{0.4x1 + 0.3x2 - 0}{130 - 0} \right] \geq d^-$$

$$(0.25) \left[\frac{x_1 - 0}{250 - 0} \right] \geq d^-$$

This is a LP problem. The solution is :

$$\underline{x} = (208.74, 82.52)$$

$$\underline{f} = (108.25, 208.74)$$

which is point E in Figs. 3.1 and 3.2.

Step 2. Decision Phase

The solutions for p = 1, 2 and oo cases are summaried in Table 3.1.

Table 3.1 Solutions Table

		f_1	f_2	x_1	x_2
	PIS	130.00	250.00	-----	-----
	NIS	0	0	-----	-----
P =	1	100.00	250.00	250.00	0.00
	2	104.00	230.00	230.00	40.00
	oo	108.25	208.74	208.74	82.52

The solutions table is presented to the DM. The DM may select one or none of the solutions.

As shown in Fig. 3.1 for p = 1, the function to be minimized $F_{p=1}$, is a linear function; therefore the preferred solution is determined by the slope of line BC and that of $F_{p=1}$. Since the slope of $F_{p=1}$ is less than that of line BC (the line representing the nondominated solutions), point C is the preferred solution. For p = 2, the function to be minimized, $F_{p=2}$, is a nonlinear (quadratic) function, and the minimum point for $F_{p=2}$ is point D in Fig. 3.1, which is the point where the $F_{p=2}$ curve touches the segment of line BC. For p = oo, the function to be minimized, $F_{p=oo}$, cannot be drawn on Fig. 3.1 but can be represented in Fig. 3.2 as point E.

Note:

Several sets of compromise solutions are obtained in Table 3.1.1 for the Hardee Company production problem by utilizing different H_p and H_n combinations.

Table 3.1.1 Solution Tables for $\underline{W}$ = (0.50, 0.50)

		f1	f2	x1	x2
	PIS	130.00	250.00	----	----
	NIS	0	0	----	----
$\underline{H}$	P	f1	f2	x1	x2
(1.00, 0.00)	1	100.00	250.00	250.00	0.00
	2	103.90	230.70	230.70	38.60
	oo	108.18	209.09	209.09	81.82
(0.75, 0.25)	1	100.00	250.00	250.00	0.00
	2	103.56	232.21	232.21	35.61
	oo	108.25	208.74	208.74	82.52
(0.50, 0.50)	1	100.00	250.00	250.00	0.00
	2	103.21	233.44	233.44	32.78
	oo	107.97	210.14	210.14	79.72
(0.25, 0.75)	1	100.00	250.00	250.00	0.00
	2	100.21	248.98	248.98	2.05
	oo	108.25	208.74	208.74	82.52
(0.00, 1.00)	1	100.00	250.00	250.00	0.00
	2	100.00	250.00	250.00	0.00
	oo	108.18	209.09	209.09	81.82

It is obvious that when parameter p equals 1, the solutions

for the problem are always the same, i.e. H has no effect on the problem's solution when p = 1. That is, the slope of Fp=1 remains constant and is less than that of line BC: therefore the preferred solutions are shown at the same point, C, in Figs 3.1 and 3.2. Almost all the solutions for p = 2 and oo are changed correspondingly when $\underline{H}$ is changed. Notice also that when $\underline{H}$ = (1.0, 0), the set of solutions is exactly the same as with the method of global criteria. (See the first numerical example of appendix B.)

3.3.2. Nutrition Problem

This problem is to determine the quantities of six foods that should be eaten to balance nutritional requirements so as to:

(i) maximize carbohydrate intake

(ii) minimize cholesterol intake

(iii) minimize cost

Information on six foods is given in Table 3.2.

Table 3.2. Nutrition and cost information on six foods

	Milk (pint)	Beef (lb.)	Eggs (dozen)	Bread (oz.)	Lettuce salad (oz.)	Orange juice (pint)	Daily intake for adults
Vitamin A (i.u.)	720	107	7080	0	134	1000	5000
Food energy (calories)	344	1460	1040	75	17.4	240	2500
Cholesterol (unit)	10	20	120	0	0	0	
Protein (g)	18	151	78	2.5	0.2	4	63
Carbohydrate(g)	24	27	0	15	1.1	52	
Iron (mg)	0.2	10.1	13.2	0.75	0.15	1.2	12.5
Cost ($)	0.22	2.2	0.8	0.1	0.05	0.26	

The problem is summarized as:

$$\max\ f_1(\underline{x}) = 24x_1 + 27x_2 + 15x_4 + 1.1x_5 + 52x_6 \quad \text{(Carbohydrate intake)}$$

$$\min\ f_2(\underline{x}) = 10x_1 + 20x_2 + 120x_3 \quad \text{(Cholesterol intake)}$$

$$\min\ f_3(\underline{x}) = 0.22x_1 + 2.2x_2 + 0.8x_3 + 0.1x_4 + 0.05x_5 + 0.26x_6 \quad \text{(Cost)}$$

Subject to:

$$720x_1 + 107x_2 + 7080x_3 + 134x_5 + 1000x_6 \geq 5000 \quad \text{(Vitamin A,i.u.)}$$

$$0.2x_1 + 10.1x_2 + 13.2x_3 + 0.75x_4 + 0.15x_5 + 1.2x_6 \geq 12.5 \quad \text{(Iron,mg)}$$

$$344x_1 + 1460x_2 + 1040x_3 + 75x_4 + 17.4x_5 + 240x_6 \geq 2500 \quad \text{(Calories)}$$

$$18x_1 + 151x_2 + 78x_3 + 2.5x_4 + 0.2x_5 + 4x_6 \geq 63 \quad \text{(Protein, g)}$$

$$x_1 \leq 6.0 \quad \text{(Milk, pints)}$$

$$x_2 \leq 1.0 \quad \text{(8eff,pounds)}$$

$$x_3 \leq 0.25 \quad \text{(Eggs, dozen)}$$

$$x_4 \leq 10.0 \quad \text{(8read, ounces)}$$

$$x_5 \leq 10.0 \quad \text{(Lettuce \& Salad,ounces)}$$

$$x_6 \leq 4.0 \quad \text{(Orange juice, pints)}$$

$$x_i \geq 0,\ i = 1, \ldots, 6$$

$$J = \{1\},\quad I = \{2, 3\}$$

Let the constraints set be denoted by $\underline{x} \in X$

The problem is solved by the following steps.

Step 0. Obtain and Construct PIS- & NIS-Payoff Tables

PIS-payoff table

	f_1	f_2	f_3	x_1	x_2	x_3	x_4	x_5	x_6
max f_1	540.00	80.00	6.09	6.00	1.00	0.00	10.00	10.00	4.00
min f_2	380.39	8.44	3.47	0.00	0.42	0.00	10.00	10.00	4.00
min f_3	281.67	67.83	2.26	3.78	0.00	0.25	10.00	0.00	0.79

$\underline{f}^* = (540.00,\ 8.44,\ 2.26)$

NIS- payoff table

	f_1	f_2	f_3	x_1	x_2	x_3	x_4	x_5	x_6
max f_1	93.34	71.53	3.08	2.50	0.83	0.25	0.00	10.00	0.00
min f_2	321.00	110.00	4.75	6.00	1.00	0.25	10.00	0.00	0.00
min f_3	540.00	110.00	6.29	6.00	1.00	0.25	10.00	10.00	4.00

$\underline{f}^- = (93.34,\ 110.00,\ 6.29)$

Step 1. Calculation phase

We assume the DM equally weights the objectives and assign $H_p = 0.75$ and $H_n = 0.25$.

Case 1. p = 1

The problem is to find $\underline{x}$ so as to:

$$\min \{ (0.75) \left[\frac{540.00 - f_1(\underline{x})}{540.00 - 93.34}\right] + (0.75) \left[\frac{f_2(\underline{x}) - 8.44}{110.00 - 8.44}\right] + (0.75) \left[\frac{f_3(\underline{x}) - 2.26}{6.29 - 2.26}\right] - (0.25) \left[\frac{f_1(\underline{x}) - 93.34}{540.00 - 93.34}\right] - (0.25) \left[\frac{110.00 - f_2(\underline{x})}{110.00 - 8.44}\right] - (0.25) \left[\frac{6.29 - f_3(\underline{x})}{6.29 - 2.26}\right] \}$$

s.t. $\underline{x} \in X$

Since all the objective functions and constraints are linear, the problem is a linear model. The solution is :

$$\underline{x} = (2.30,\ 0.00,\ 0.00,\ 10.00,\ 0.00,\ 4.00)$$

$$\underline{f} = (413.12,\ 22.97,\ 2.55).$$

Case 2. p = 2

The problem becomes to find $\underline{x}$ so as to:

$$\min \{ (0.75) \left[\frac{540.00 - f_1(\underline{x})}{540.00 - 93.34}\right]^2 + (0.75) \left[\frac{f_2(\underline{x}) - 8.44}{110.00 - 8.44}\right]^2 + (0.75) \left[\frac{f_3(\underline{x}) - 2.26}{6.29 - 2.26}\right]^2 - (0.25) \left[\frac{f_1(\underline{x}) - 93.34}{540.00 - 93.34}\right]^2 - (0.25) \left[\frac{110.00 - f_2(\underline{x})}{110.00 - 8.44}\right]^2 - (0.25) \left[\frac{6.29 - f_3(\underline{x})}{6.29 - 2.26}\right]^2 \}$$

$$\text{s.t. } \underline{x} \in X$$

This is a nonlinear problem. We use SUMT to solve it. The solution is :

$$\underline{x} = (1.69,\ 0.09,\ 0.01,\ 9.99,\ 3.61,\ 4.00)$$

$$\underline{f} = (404.83,\ 19.68,\ 2.81).$$

Case 3. p = oo

The problem is to find $\underline{x}$ so as to:

$$\min \; (d^{*} - d^{-})$$

$$\text{s.t.} \quad \underline{x} \in X$$

$$(0.75) \left[\frac{540.00 - f_1(\underline{x})}{540.00 - 93.34} \right] \leq d^{*}$$

$$(0.75) \left[\frac{f_2(\underline{x}) - 8.44}{110.00 - 8.44} \right] \leq d^{*}$$

$$(0.75) \left[\frac{f_3(\underline{x}) - 2.26}{6.29 - 2.26} \right] \leq d^{*}$$

$$(0.25) \left[\frac{f_1(\underline{x}) - 93.34}{540.00 - 93.34} \right] \geq d^{-}$$

$$(0.25) \left[\frac{110.00 - f_2(\underline{x})}{110.00 - 8.44} \right] \geq d^{-}$$

$$(0.25) \left[\frac{6.29 - f_3(\underline{x})}{6.29 - 2.26} \right] \geq d^{-}$$

This is a LP problem. The solution is :

$$\underline{x} = (3.08, \; 0.00, \; 0.00, \; 10.00, \; 8.68, \; 4.00)$$

$$\underline{f} = (441.47, \; 30.80, \; 3.15)$$

Step 2. Decision Phase

The solutions for p = 1,2 and oo cases are summarized in Table 3.3.

Table 3.3 Solutions Table

	f_1	f_2	f_3	x_1	x_2	x_3	x_4	x_5	x_6
PIS	540.00	8.44	2.26	---	---	---	---	---	---
NIS	93.34	110.00	6.29	---	---	---	---	---	---
p = 1	413.12	22.97	2.55	2.30	0.00	0.00	10.00	0.00	4.00
2	404.83	19.68	2.81	1.69	0.09	0.01	9.99	3.61	4.00
oo	441.47	30.80	3.15	3.08	0.00	0.00	10.00	8.68	4.00

The solutions table is presented to the DM for evaluation. The DM may select one or none of the solutions.

3.3.3 Bow River Valley Water Quality Management Problem

This problem is based on a hypothetical case developed by Dorfman and Jacoby [1969] and modified by Monarchi et al. [1973]. It is concerned with the pollution problems of a river basin, the Bow River Valley, whose main features are shown in Fig. 3.3. Industrial pollution originates from the Pierce-Hall Cannery, located near the head of the valley, and municipal waste pollution comes from two outlets at Bowville and Plympton. A state park is located between the cities, and the lower end of the valley is a part of the state boundary line.

The Bow Valley Water Pollution Control Commission is made up of representatives from all three waste dischargers and members of the state and federal government. The commission is responsible for maintaining river water quality by setting the waste reduction requirements at the three sources of pollution, but it must act with an awareness of the effect of any additional effluent treatment costs on the economy of the valley.

The Commission (the DM) is concerned with the following six objectives: the DO levels at Bowville, Robin State Park, and Plympton (objectives 1, 2, and 3, respectively); the percent return on investment at the Pierce-Hall Cannery (objective 4); and the addition to the tax rate for Bowville and Plympton (objectives 5 and 6, respectively). The first problem of the DM is to determine a waste treatment policy which satisfies the constraint that the DO level at the state line is above 3.5 mg/l.

The problem is summarized as follows: determine x_1, x_2 and x_3 so as to satisfy the following objectives, constraints and goals.

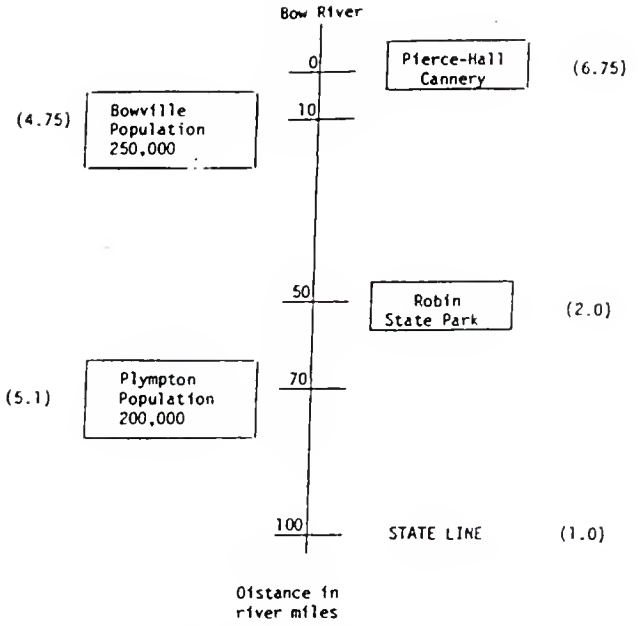

Figure 3.3 Main Features of the Bow River Valley

The six objectives are:

$$\max \quad f_1 = 4.75 + 2.27 \ (x_1 - 0.3) \quad (\text{mg/l DO})$$

$$\max \quad f_2 = 2.0 + 0.524 \ (x_1 - 0.3) + 2.79 \ (x_2 - 0.3) + 0.882 \ (w_1 - 0.3) + 2.65 \ (w_2 - 0.3) \ (\text{mg/l DO})$$

$$\max \quad f_3 = 5.1 + 0.177 \ (x_1 - 0.3) + 0.978 \ (x_2 - 0.3) + 0.216 \ (w_1 - 0.3) + 0.768 \ (w_2 - 0.3) \ (\text{mg/l DO})$$

$$\max \quad f_4 = 7.5 - 0.012 \ \left(\frac{59}{1.09 - x_1^2} - 59 \right) \ (\text{\% return})$$

$$\min \quad f_5 = 1.80 * 10^{-3} \ \left(\frac{532}{1.09 - x_2^2} - 532 \right) (\text{\% additional tax rate})$$

$$\min \quad f_6 = 2.50 * 10^{-3} \ \left(\frac{450}{1.09 - x_3^2} - 450 \right) (\text{\% additional tax rate})$$

$$\text{s.t.} \quad g_1 = 1.0 + 0.0332(x_1 - 0.3) + 0.0186(x_2 - 0.3) + 3.34(x_3 - 0.3) + 0.0204(w_1 - 0.3) + 0.778(w_2 - 0.3) + 2.62(w_3 - 0.3) \geq 3.5$$

$$0.3 \leq x_i \leq 1.0, \qquad i = 1, 2, 3 \qquad (3.21)$$

Let the above constraints be denoted by $\underline{x} \in X$.

w_i is related to x_i by:

$$w_i = \frac{0.39}{1.39 - x_i^2}, \qquad i = 1,2,3$$

The original problem formulation is slightly modified to reduce its complexity. Both objectives corresponding to DO levels at Bowville and Plymton are changed to constraints for DO $\geq$ 6.0. The modified problem is given below.

$$\max f_1(\underline{x}) = 2.0 + 0.524(x_1 - 0.3) + 2.79(x_2 - 0.3) \qquad (3.22)$$
$$+ 0.882(w_1 - 0.3) + 2.65(w_2 - 0.3) \text{ (mg/l DO, at state park)}$$

$$\max f_2(\underline{x}) = 7.5 - 0.012 \left(\frac{59}{1.09 - x_1^2} - 59 \right) \text{ (\% earning, for cannery)}$$

$$\min f_3(\underline{x}) = 1.80 * 0.001 \left(\frac{532}{1.09 - x_2^2} - 532 \right) \text{ (\$/\$1000 additional tax at Bowville)}$$

$$\min f_4(\underline{x}) = 2.50 * 0.001 \left(\frac{450}{1.09 - x_3^2} - 450 \right) \text{ (\$/\$1000 additional tax at Plymton)}$$

$$\text{s.t. } g_1(\underline{x}) = 4.75 + 2.27 (x_1 - 0.3) \geq 6.0$$

$$g_2(\underline{x}) = 5.1 + 0.177 (x_1 - 0.3) + 0.978(x_2 - 0.3)$$
$$+ 0.216(w_1 - 0.3) + 0.768(w_2 - 0.3) \geq 6.0$$

$$g_3(\underline{x}) = 1.0 + 0.0332(x_1 - 0.3) + 0.0186 (x_2 - 0.3)$$
$$+ 3.34(x_3 - 0.3) + 0.0204(w_1 - 0.3)$$
$$+ 0.778(w_2 - 0.3) + 2.62(w_3 - 0.3) \geq 3.5$$

$$0.3 \leq x_i \leq 1.0, \qquad i = 1, 2, 3$$

Let the above constraints be denoted by $\underline{x} \in X$. w_i is related to x_i by:

$$w_i = \frac{0.39}{1.39 - x_i^2} \quad . \qquad i = 1,2,3$$

We solve the problem by the following steps.

Step 0. Obtain and construct PIS- & NIS-payoff tables

PIS - Payoff Table

	f_1	f_2	f_3	f_4	x_1	x_2	x_3
f_1	6.7922	.3413	9.6824	1.2110	.9999	.9999	.7800
f_2	5.2777	6.2752	2.6735	1.9306	.8507	.9090	.8496
f_3	4.8599	.3413	1.0418	2.1468	.9999	.7817	.8638
f_4	6.7911	.3761	9.6824	.9588	.9999	1.0000	.7417

$\underline{f}^* = (6.7922, 6.2752, 1.0418, 0.9588)$

NIS - Payoff Table

	f_1	f_2	f_3	f_4	x_1	x_2	x_3
f_1	4.7843	6.1099	1.5173	2.0793	.8675	.8385	.8596
f_2	5.4931	.3413	2.0919	1.9794	.9999	.8809	.8530
f_3	6.4856	5.4252	9.6824	1.6066	.9141	.9999	.8235
f_4	5.3276	5.4192	2.4319	11.3750	.9144	.8986	1.0000

$\underline{f}^- = (4.7843, 0.3413, 9.6824, 11.3750)$

Step 1. Calculating phase

We assume the four objectives are of equal importance, i.e. $\underline{w}$ = (1.0, 1.0, 1.0, 1.0), and H = (0.5, 0.5). We also assume the analyst wants only the p = 1, 2 and oo cases.

The distance function for measuring the closeness to the positive ideal solution is given as:

$$F_p = w_1 \left[\frac{6.7922 - f_1(\underline{x})}{(6.7922 - 4.7843)}\right]^p + \qquad (3.23)$$

$$w_2 \left[\frac{6.2725 - f_2(\underline{x})}{(6.2725 - 0.3413)}\right]^p +$$

$$w_3 \left[\frac{f_3(\underline{x}) - 1.0418}{(9.6824 - 1.0418)}\right]^p +$$

$$w_4 \left[\frac{f_4(\underline{x}) - 0.9588}{(11.3750 - 0.9588)}\right]^p$$

The distance function for measuring the distance to negative idea solution.

$$N_p = w_1 \left[\frac{f_1(\underline{x}) - 4.7843}{(6.7922 - 4.7843)}\right]^p + \qquad (3.24)$$

$$w_2 \left[\frac{f_2(\underline{x}) - 0.3413}{(6.2725 - 0.3413)}\right]^p +$$

$$w_3 \left[\frac{9.6824 - f_3(\underline{x})}{(9.6824 - 1.0418)}\right]^p +$$

$$w_4 \left[\frac{11.3750 - f_4(\underline{x})}{(11.3750 - 0.9588)}\right]^p$$

Case 1. p = 1

The problem is to find $\underline{x}$ so as to:

$$\min \{ H_p F_1 - H_n N_1 \}$$

$$\text{s.t.} \quad \underline{x} \in X$$

This is a nonlinear programming problem. We use Sequential Unconstrainted Minimization Technique (SUMT) to solve the problem. The solution is:

$\underline{x}$ = (0.8507, 0.9476, 0.7726)

$\underline{f}$ = (5.5624, 6.2752, 4.0285, 1.1565)

Case 2. p = 2

The problem is to find $\underline{x}$ so as to:

$$\min \{ H_p F_2 - H_n N_2 \}$$

$$\text{s.t.} \quad \underline{x} \in X$$

This is a nonlinear programming problem. We use SUMT to solve it. The solution is given as:

$\underline{x}$ = (0.8507, 0.9484, 0.7722)

$\underline{f}$ = (5.6611, 6.2752, 4.0682, 1.1540)

Case 3. p = oo

The problem is not solvable with the current problem formulation. We rewrite (3.23) as:

min d^*

s.t. $\underline{x} \in X$

$$w_1 \left[\frac{6.7922 - f_1(\underline{x})}{6.7922 - 4.7843} \right] \leq d^* \qquad (3.25)$$

$$w_2 \left[\frac{6.2725 - f_2(\underline{x})}{6.2725 - 0.3413} \right] \leq d^*$$

$$w_3 \left[\frac{f_3(\underline{x}) - 1.0418}{9.6824 - 1.0418} \right] \leq d^*$$

$$w_4 \left[\frac{f_4(\underline{x}) - 0.9588}{11.3750 - 0.9588} \right] \leq d^*$$

Similarly, we rewrite (3.24) as:

max d^-

s.t. $\underline{x} \in X$

$$w_1 \left[\frac{f_1(\underline{x}) - 4.7843}{6.7922 - 4.7843'} \right] \geq d^- \qquad (3.26)$$

$$w_2 \left[\frac{f_2(\underline{x}) - 0.3413}{6.2725 - 0.3413} \right] \geq d^-$$

$$w_3 \left[\frac{9.6824 - f_3(\underline{x})}{9.6824 - 1.0418} \right] \geq d^-$$

$$w_4 \left[\frac{11.3750 - f_4(\underline{x})}{11.3750 - 0.9588} \right] \geq d^-$$

Combination of (3.25) and (3.26) gives a solvable problem :

$$\min (d^* - d^-) \qquad (3.27)$$

s.t. $\underline{x} \in X$

$$H_p \, w_1 \left[\frac{6.7922 - f_1(\underline{x})}{6.7922 - 4.7843} \right] \leq d^*$$

$$H_p \, w_2 \left[\frac{6.2725 - f_2(\underline{x})}{6.2725 - 0.3413} \right] \leq d^*$$

$$H_p \, w_3 \left[\frac{f_3(\underline{x}) - 1.0418}{9.6824 - 1.0418} \right] \leq d^*$$

$$H_p \, w_4 \left[\frac{f_4(\underline{x}) - 0.9588}{11.3750 - 0.9588} \right] \leq d^*$$

$$H_n \, w_1 \left[\frac{f_1(\underline{x}) - 4.7843}{6.7922 - 4.7843} \right] \geq d^-$$

$$H_n\, w_2 \left[\frac{f_2(\underline{x}) - 0.3413}{6.2725 - 0.3413}\right] \geq d^-$$

$$H_n\, w_3 \left[\frac{9.6824 - f_3(\underline{x})}{9.6824 - 1.0418}\right] \geq d^-$$

$$H_n\, w_4 \left[\frac{11.3750 - f_4(\underline{x})}{11.3750 - 0.9588}\right] \geq d^-$$

The problem is solved by SUMT. The solution is given as :

$\underline{x}$ = (0.9608, 0.9547, 0.8268)

$\underline{f}$ = (6.0098, 3.9650, 4.4057, 1.6432)

Step 2. Evaluation phase

The three sets of solutions are summarized in Table 3.4. The DM may select one or none of them.

Table 3.4 Solutions Table

	f1	f2	f3	f4	x1	x2	x3
PIS	6.7922	6.2725	1.0418	0.9588	----	----	----
NIS	4.7843	0.3413	9.6824	11.3750	----	----	----
p=1	5.5624	6.2752	4.0285	1.1565	0.8507	0.9476	0.7726
2	5.6611	6.2752	4.0682	1.1540	0.8507	0.9484	0.7722
oo	6.0098	3.9650	4.4057	1.6432	0.9608	0.9547	0.8268

3.4 Remarks

(1) TOPSIS-MODM requires no subjective preference information from the DM once the problem constraints and objectives have been defined. Thus this method may require the DM to accept the solution obtained from the method.

(2) The solutions obtained in the calculation phase depend not only on the constraints and objectives but also heavily rely on the parameter p. Notice that in our numerical examples, no solutions are identical in any given solution table. That is the parameter p locates these solution points which are geometrically different from one another.

(3) Unlike those methods that generate a whole set of nondominated solutions which take the DM a lot of time to process and absorb, TOPSIS-MODM generates only three nondominated solutions. This feature makes the selection of the preferred solution less time-consuming and easier.

Chapter 4 Interactive Sequential Goal Programming II (ISGP II)

4.1 Introduction

A new approach called Interactive Sequential Goal Programming II (ISGP II) is introduced, which is an updated ISGP of Masud and Hwang [1981]. It combines the attractive features of the following MODM methods: STEM method's [Benayoun, et al., 1971] positive-ideal solution concept and the interactive calculation phase and decision phase process between the DM and the analyst; Charnes and Cooper's [1961] original Goal Programming concept; SEMOPS's [Monarchi, et al., 1976] principal problem and auxiliary problems approach. In ISGP II, the new concepts of negative-ideal solutions of multiple objectives and a new normalization process of each objective by incorporating the positive-ideal solution and negative-ideal solution are introduced. The Maximum Achievable Rate (MAR) and the Maximum Achievable Goal (MAG) for the multiple objectives are also introduced. The set of MAG is a feasible solution which has the highest same rate, MAR, for all objectives.

Maximum Achievable Rate (MAR) and Maximum Achievable Goal (MAG): In a group decision making problem, if each objective represents one decision maker's payoff, then the positive-ideal solution, $\underline{f}^*$, if attainable, would make each DM happy because it simultaneously optimizes each DM's payoff. However, in a real world problem, $\underline{f}^*$ is usually not attainable. To dissolve the group conflict, a compromise among the DM's is necessary. That

is, each DM can expect to achieve only portion of the f^* instead of the entire f^*. The amount of sacrifice by each DM must be equal in order to maintain a "cooperative group spirit". Hence, MAR and MAG are introduced to satisfy that condition.

MAG and MAR are determined by the analysts prior to having the DM set initial goals, therefore they served as excellent reference values. The MAG may be interpreted as a set of nondominated solutions which is the individual DM's largest amount of resources s/he can have without the sacrifices of the other DMs. MAR is the maximum percentage of resources a DM can have without jeopardizing the group harmony. There is a procedure (see ISGP II algorithm for details) which operates in an iterative fashion that can be used to derive the values of MAG and MAR.

With the concept of MAR and MAG, we proceed to formulate the algorithm of ISGP II.

4.2 Algorithm For ISGP II

Problem Formulation:

The MODM problem is:

$$\text{Max/Min } [\, f_1(\underline{x}),\ f_2(\underline{x}),\ldots,\ f_k(\underline{x})] \tag{4.1}$$

$$\text{s.t. } \underline{x} \in X$$

where $f_j(\underline{x})$: the benefit objective for maximization, $j \in J$

$f_i(\underline{x})$: the cost objective for minimization, $i \in I$

Algorithm:

The complete ISGP II algorithm for this problem is given below.

Step 0 Determine Positive Ideal Solution (PIS), $\underline{f}^*$, and Negative Ideal Solution (NIS), $\underline{f}^-$.

Step 0-1 Obtain and Construct PIS - Payoff Table.

The problem is to find $\underline{x}$ so as to

$$\max\ f_j(\underline{x}),\ j \in J \tag{4.2}$$

$$\text{s.t. } \underline{x} \in X$$

and / or

$$\min\ f_i(\underline{x}),\ i \in I \tag{4.3}$$

$$\text{s.t. } \underline{x} \in X$$

The solution of (4.2), $\underline{x}^*$ and f_j^*, is known as the positive-ideal solution. Similarly, the solution of (4.3), $\underline{x}^*$ and f_i^*, is

also known as the positive-ideal solution. With all the $\underline{x}^*$ from (4.2) and (4.3), the PIS-Payoff Table is given as:

PIS - Payoff Table

	f_1	f_2	...	f_k
f_1	f_1^*	f_{12}	...	f_{1k}
f_2	f_{21}	f_2^*	...	f_{2k}
.				
.				
.				
f_k	f_{k1}	f_{k2}	...	f_k^*

$$\underline{f}^* = (f_1^*, f_2^*, \ldots, f_k^*)$$

Step 0-2 Obtain and Construct NIS-Payoff Table.

The problem is to find $\underline{x}$ so as to

$$\min f_j(\underline{x}), \quad j \in J \tag{4.4}$$
$$\text{s.t.} \quad \underline{x} \in X$$

and / or

$$\max f_i(\underline{x}), \quad i \in I \tag{4.5}$$
$$\text{s.t.} \quad \underline{x} \in X$$

The solutions of (4.4) and (4.5), $\underline{x}^-$ and f_r^-, are known as the negative-ideal solutions. With all the $\underline{x}^-$ from (4.4) and (4.5), the NIS-Payoff Table is constructed as:

NIS-Payoff Table

	f_1	f_2	...	f_k
f_1	f_1^-	z_{12}	...	z_{1k}
f_2	z_{21}	f_2^-	...	z_{2k}
.				
.				
.				
f_k	z_{k1}	z_{k2}	...	f_k^-

$$\underline{f}^- = (f_1^-, f_2^-, \ldots, f_k^-)$$

Note that $\underline{f}^*$ and $\underline{f}^-$ provide the upper and lower bounds on the objective function values and are a great source of information for the DM in all the following steps.

Step 1 Set initial goals.

Step 1.1 Obtain a set of Maximum Achievable Goals (MAG) and a Maximum Achievable Rate (MAR), B%. (The MAG is a feasible solution which has the highest same rate, MAR, for all objectives.)

The problem is to find $\underline{x}$ so as to

$$\min a^g = \{ \sum_{r=1}^{k} W_r \, d_r^- \} \tag{4.6}$$

s.t. $\underline{x} \in X$

$$f_j(\underline{x}) + (d_j^- - d_j^+) (f_j^* - f_j^-) = b_j^*, \; j \in J \tag{4.6a}$$

$$f_i(\underline{x}) - (d_i^- - d_i^+) (f_i^- - f_i^*) = b_i^*, \; i \in I \tag{4.6b}$$

$$0 \leq d_r^- \leq 1, \; r = 1,2,\ldots,k$$

where $b_j^* = f_j^* - (1 - B\%)(f_j^* - f_j^-)$, $j \in J$

$b_i^* = f_i^* + (1 - B\%)(f_i^- - f_i^*)$, $i \in I$

The solution is given as $\underline{x}$, $\underline{f}^g$, and a^g.

Note that (4.6a) and (4.6b) are, respectively, transformed from

$$\frac{f_j^* - f_j(\underline{x})}{f_j^* - f_j^-} - (d_j^- - d_j^+) = \frac{f_j^* - b_j^*}{f_j^* - f_j^-}, \quad j \in J$$

and

$$\frac{f_i(\underline{x}) - f_i^*}{f_i^- - f_i^*} + (d_i^+ - d_i^-) = \frac{b_i^* - f_i^*}{f_i^- - f_i^*}, \quad i \in I$$

Also note that MAR is unknown initially. To properly start the calculation of (4.6), we have to assign a value to B%. Since we perceive that when B% is set at 75% we can obtain a MAR with minimum effort, we suggest B% be set at 75% to start the calculation of (4.6). Steps 1.1.1 and 1.1.2 illustrate the details.

Step 1.1.1 Check if $\underline{f}^g$ is a set of nondominated solutions.

If $a^g = 0$, go to step 1.1.2.

If $a^g \neq 0$, we decrease B% by 5%, i.e. B% = (B - 5)%, recalculate (4.6), and check the value of a^g again. The procedure is repeated until an answer $a^g = 0$ is found. After an $a^g = 0$ is found, we increase B% by 1%, i.e. B% = (B + 1)%, recalculate (4.6), and check the value of a^g again. The procedure repeats

until an $a^g \neq 0$ is found, and that $a^g \neq 0$ signals that MAG and MAR are found. MAR is the highest rate for all objectives in (4.6) for which a feasible nondominated solution is obtainable; this is MAG. Go to step 1.2 .

Step 1.1.2

When $a^g = 0$, we increase B% by 5%, i.e. B% = (B + 5)%, recalculate (4.6), and check the value of a^g again. The procedure is repeated until an answer $a^g \neq 0$ is found. After an $a^g \neq 0$ is obtained, we decrease B% by 1%, i.e. B% = (B - 1)%, recalculated (4.6), and check the value of a^g again. The procedure is repeated until an answer $a^g = 0$ is found, and that $a^g = 0$ signals that MAG and MAR is found. We conclude that MAR is (B + 1)% and MAG is a set of nondominated solutions obtained at (B + 1)%. Go to Step 1.2. Notice that MAG and MAR are used as another important reference in all the following steps.

Step 1.2 Display ISGP II-Initial Table.

	max/min $f_1(\underline{x})$	max/min $f_2(\underline{x})$	...	max/min $f_k(\underline{x})$
$\underline{f}^*$	f_1^*	f_2^*	...	f_k^*
MAG (B%)	b_1^*	b_2^*	...	b_k^*
$\underline{f}^-$	f_1^-	f_2^-	...	f_k^-
$\underline{w}$	w_1	w_2	...	w_k

The initial table is presented to the DM. If the DM is satisfied with the MAG then take it as the preferred solution and

go to step 8. Otherwise go to step 1.3 .

Step 1.3 Set initial goals.

The DM set his/her initial goals at $\underline{b}$ referring to the initial table. The initial goals, $\underline{b}$, should satisfy

$$f_j^- < b_j \leq f_j^*, \quad j \in J \tag{4.7}$$

$$f_i^* \leq b_i < f_i^-, \quad i \in I$$

Step 2 Solve for PS (Principal Solution).

The goal constraints associated with the objectives are:

$$\frac{f_j^* - f_j(\underline{x})}{f_j^* - f_j^-} - (d_j^- - d_j^+) = \frac{f_j^* - b_j}{f_j^* - f_j^-}, \quad j \in J$$

and (4.8)

$$\frac{f_i(\underline{x}) - f_i^*}{f_i^- - f_i^*} + (d_i^+ - d_i^-) = \frac{b_i - f_i^*}{f_i^- - f_i^*}, \quad i \in I$$

such that (4.7) will not be violated.

Expression (4.8) can be rewritten as:

$$f_j(\underline{x}) + (d_j^- - d_j^+)(f_j^* - f_j^-) = b_j, \quad j \in J \tag{4.9}$$

$$f_i(\underline{x}) - (d_i^- - d_i^+)(f_i^- - f_i^*) = b_i, \quad i \in I$$

Step 2.1 Calculation of PS (Principal Solution).

The problem is to find a set of $\underline{x}$ so as to

$$\min a_{ps} = \{ \sum_{r=1}^{k} W_r \, d_r^- \} \tag{4.10}$$

$$\text{s.t.} \quad \underline{x} \in X$$

$$f_j(\underline{x}) + (d_j^- - d_j^+)(f_j^* - f_j^-) = b_j, \; j \in J$$

$$f_i(\underline{x}) - (d_i^- - d_i^+)(f_i^- - f_i^*) = b_i, \; i \in I$$

$$0 \leq d_r^-, \; d_r^+ \leq 1$$

where W_r is the relative importance of each objective assigned by the DM.

Solution: $\underline{x}^o$, $\underline{f}^o$, a_{ps}

d_r^- is the underachievement of the objective function $f_r(\underline{x})$ from its goal b_r. $\Sigma W_r \, d_r^-$ equals zero if the goal is in the feasible region while $\Sigma W_r \, d_r^-$ is greater than zero if the goal is beyond the feasible region.

Step 2.2 Check if $\underline{f}^o$ is a set of nondominated solutions

If $a_{ps} = 0$ then go to step 2.3.

If $a_{ps} \neq 0$ then take $\underline{f}^o$ as the principal solution and go to step 3.

Step 2.3 Improvement of $\underline{f}^o$.

$a_{ps} = 0$ implies that the $\underline{f}^o$ obtained is a set of dominated solutions. We increase b_j and decrease b_i by z%, recalculate

(4.10), and check the value of a_{ps}. The procedure repeats until an $a_{ps} \neq 0$ is found. We recommed z to be set at 5, 10, 15, or 20% of the goal of each objective with a default of 5%. Notice that the goal setting rules in (4.7) must be satisfied when modifying b_j and b_i.

Step 3 Solve for k sets of ASs (Auxiliary Solutions).

Each set of auxiliary solutions is obtained by setting one individual goal as a constraint and by minimizing the summarized weighted deviations from other goals.

The problem is to find a set of $\underline{x}$ so as to

$$\min a^u = \{ \sum_{\substack{r=1 \\ r \neq u}}^{k} W_r \, d_r^- \} \qquad (4.11)$$

$$\text{s.t.} \quad \underline{x} \in X$$

$$f_j(\underline{x}) + (d_j^- - d_j^+)(f_j^* - f_j^-) = b_j, \text{ (for all } j, j \neq u)$$

$$f_i(\underline{x}) - (d_i^- - d_i^+)(f_i^- - f_i^*) = b_i, \text{ (for all } i, i \neq u)$$

$$f_u(\underline{x}) \geq b_u \quad (\text{if } u \in J)$$

or

$$f_u(\underline{x}) \leq b_u \quad (\text{if } u \in I)$$

$$0 \leq d_r^- \leq 1 \qquad (\text{for all } r, r \neq u)$$

where W_r is the relative importance of each objective assigned by the DM.

Solution: $\underline{x}^u, \underline{f}^u, a^u, u = 1, 2, \ldots, k$

Step 3.1 Check if $\underline{f}^u$ is a set of nondominated solutions.

If $a^u = 0$ then go to step 3.2.

If $a^u \neq 0$, then we take $\underline{f}^u$ as the u^{th} auxiliary solution and go back to step 3 for the $(u + 1)^{th}$ auxiliary solution. If all k sets of auxiliary solutions are obtained, i.e. $u + 1 > k$, we proceed to step 4.

Step 3.2 Improvement of $\underline{f}^u$.

$a^u = 0$ implies that the previously obtained $\underline{f}^u$ is a set of dominated solutions. We increase b_j and decrease b_i by z%, recalculate (4.11), and check the value of a^u again. The procedure repeats until an $a^u \neq 0$ is found, i.e. a set of nondominated solutions is obtained. Z% has the same featuresas those in step 2.3 . Notice that the goal setting rules in (4.7) must be satisfied when modifying b_j and b_i. Go to step 3.

Step 4 Form ISGP II-Tradeoff Table.

The tradeoff table is given in Table 4.1. The first row marked $\underline{f}^*$ represents the positive-ideal solutions. The second row marked MAG (B%) indicates the maximum achievable goal and the maximum achievable rate. The third row marked $\underline{f}^-$ represents the negative ideal. The fourth row marked $\underline{w}$ indicates the weights of each objective assigned by the DM. All four rows appear on the tradeoff table for each iteration. The notation "#" right below dotted line indicates the number of iterations. The fifth row marked Goal indicates the goals set by the DM. The Sixth row

marked PS indicates the current best nondominated solution. Each of the remaining rows indicates a different AS.

Table 4.1 ISGP II-Tradeoff Table

	max/min $f_1(\underline{x})$	max/min $f_2(\underline{x})$	...	max/min $f_k(\underline{x})$
$\underline{f}^*$	f_1^*	f_2^*	...	f_k^*
MAG (B%)	b_1^*	b_2^*	...	b_k^*
$\underline{f}^-$	f_1^-	f_2^-	...	f_k^-
$\underline{w}$	w_1	w_2	...	w_k
Iteration #				
goal	b_1	b_2	...	b_k
PS	f_1^0	f_2^0	...	f_k^0
AS1	b_1	f_2^1	...	f_k^1
AS2	f_1^2	b_2	...	f_k^2
:				
ASK	f_1^k	f_2^k	...	b_k

Step 5 Evaluation phase.

Step 5.1 Ask the DM:

Is $\underline{f}^0$ or any $\underline{f}^u$ satisfactory ?

If yes, then go to step 8, otherwise go to step 5.2.

Step 5.2 Ask the DM to specify his/her new goals, $\underline{b}$, from the information in the tradeoff table.

Step 6 Calculate the deviations of the new goals from PS.

$$\hat{f}_j^- = f_j^o - b_j, \text{ for all } j = \{ j \mid b_j \leq f_j^o \}$$

$$\hat{f}_j^+ = b_j - f_j^o, \text{ for all } j = \{ j \mid b_j > f_j^o \}$$

$$\hat{f}_i^- = b_i - f_i^o, \text{ for all } i = \{ i \mid b_i \geq f_i^o \}$$

$$\hat{f}_i^+ = f_i^o - b_i, \text{ for all } i = \{ i \mid b_i < f_i^o \}$$

Step 7 Consistency check for the specified $\underline{b}$.

For all $r = 1,2,\ldots,k$, check to see if there is at least one $\hat{f}_r^- > 0$ when one or more $\hat{f}_r^+ > 0$.

If yes, then go to step 2.1, otherwise go to step 5.2 . Since $\underline{f}^o$ is a nondominated solution, this consistency check is necessary to make sure that if the DM wants an improvement in one of the objective values from $\underline{f}^o$, s/he is ready to accept a detriment in at least one other objective value.

Step 8 Stop.

Print the preferred solution $\underline{x}$ and $\underline{f}$ and terminate the algorithm.

Note that in solving (4.6), (4.10), and (4.11), any appropriate single objective optimization technique can be utilized iteratively.

4.3 Numerical Examples

4.3.1 Hardee Company Production Scheduling.

The Hardee company makes two kinds of dolls. Doll A is a high quality doll and Doll B is of lower quality. The respective profits are \$0.40 and \$0.30 per doll. Each Doll A requires twice as much labor as a Doll B, and if all dolls were of type B, the company could make 500 per day. The supply of material is sufficient for only 400 dolls per day (both A and B combined). The problem assumes that all the dolls for type A and type B the factory can make could be sold, and that the best customer of the company wishes to have as many as possible of type A doll. The manager realizes that two objectives: (1) the maximization of profit, and (2) the maximization of production of Doll A, should be considered in scheduling the production.

This MODM problem can be formulated as follows:

$$\max f_1(\underline{x}) = 0.4x_1 + 0.3x_2$$

$$\max f_2(\underline{x}) = x_1$$

subject to

$$g_1(\underline{x}) = x_1 + x_2 \leq 400$$

$$g_2(\underline{x}) = 2x_1 + x_2 \leq 500$$

$$x_1, x_2 \geq 0$$

where x_1 and x_2 are the number of Doll A and that of Doll B produced respectively. Let constraints be denoted by $\underline{x} \in X$.

The problem is solved by the following steps.

Step 0 Obtain and Construct PIS & NIS Payoff Tables.

We solve max $f_1(\underline{x})$ and max $f_2(\underline{x})$ subject to the constraints, respectively, and obtain the PIS-Payoff Table as:

	f_1	f_2
f_1	130^*	100
f_2	100	250^*

$$\underline{f}^* = (130, 250)$$

Similarly, we solve min $f_1(\underline{x})$ and min $f_2(\underline{x})$ subject to $\underline{x}$ X, respectively, and obtain the NIS-Payoff Table as:

	f_1	f_2
f_1	0^-	0
f_2	0	0^-

$$\underline{f}^- = (0, 0)$$

Step 1.1 Obtain a set of MAG and MAR.

We (the analysts) set B % = 75%, and solve the following problem:

$$\min a^g = \{ d_1^- + d_2^- \}$$

$$\text{s.t. } \underline{x} \in X \qquad (4.5)$$

$$f_1(\underline{x}) + 130\ (d_1^- - d_1^+) = b_1$$

$$f_2(\underline{x}) + 250\ (d_2^- - d_2^+) = b_2$$

$$0 \leq d_r^-\ ,\ d_r^+ \leq 1\ ,\ r = 1,\ 2$$

where $b_1 = 130 - (1 - 75\%) (130 - 0) = 97.50$

$b_2 = 250 - (1 - 75\%) (250 - 0) = 187.50$

This is a linear programming problem. The solution is given as:

$\underline{x} = (187.50,\ 75.00)$

$\underline{f}^g = (97.50,\ 187.50)$

$a^g = 0$

The answer $a^g = 0$ implies that with the given $\underline{b}$ there exists a dominated $\underline{f}^g$ which is not appropriate to serve as MAG.

Since B% = 75% results in a dominated $\underline{b}$ and $\underline{f}^g$, we increase B% by 5%, i.e. B% = 80%, and recalculate (4.5) with

$b_1 = 130 - (1 - 80\%) (130 - 0) = 104.00$

$b_2 = 250 - (1 - 80\%) (250 - 0) = 200.00$

The solution is given as

$\underline{x} = (200.00,\ 80.00)$

$\underline{f}^g = (104.00,\ 200.00)$

$a^g = 0$

Again, we find the new $a^g = 0$, i.e. $\underline{f}^g$ is a set of dominated solutions. We can increase B% by another 5%, i.e. B% = 85%. Another calculation of (4.5) is conducted with

$$b_1 = 130 - (1 - 85\%) (130 - 0) = 110.50$$

$$b_2 = 250 - (1 - 85\%) (250 - 0) = 212.50$$

The solution is given as:

$$\underline{x} = (212.50, 75.00)$$

$$\underline{f}^g = (107.50, 212.50)$$

$$a^g = 0.0231$$

The answer $a^g \neq 0$ implies that $\underline{f}^g$ is a nondominated solution. The MAR, B%, is located somewhere between 85% and 81%. To locate the precise B%, we decrease B% by 1%, i.e. B% = 84%, and calculate (4.5) with $b_1 = 109.20$ and $b_2 = 210.00$.

The solution is given as

$$\underline{x} = (210.00, 80.00)$$

$$\underline{f}^g = (108.00, 210.00)$$

$$a^g = 0.0092$$

Since the answer a^g is not zero, we decrease B% by another 1%, i.e. B% = 83%, and proceed to another calculation of (4.5) with $\underline{b}$ = (107.90, 207.50). The solution is given as

$$\underline{x} = (207.50, 83.00)$$

$$\underline{f}^g = (107.90, 207.50)$$

$$a^g = 0$$

The answer $a^g = 0$ implies that the $\underline{f}^g$ when b = 83% is a set of dominated solutions. We conclude that MAG is located at $\underline{f}^g$ = (108.00, 210.00) and MAR is 84%, i.e. B% = 84%.

The scheme for obtaining MAG and MAR is depicted in figure 4.1. Points P and N denote the positive-ideal solution (130.00, 250.00) and negative-ideal solution (0, 0), respectively. The heavy solid line AB represents the nondominated solution region. Capital S denotes the feasible region for our problem. Point 1, $\underline{f}$ = (97.50, 187.50), is the set of solutions when B% = 75% (or $\underline{b}$ = 97.50, 187.50)). Point 2, $\underline{f}$ = (104.00, 200.00), is the set of solutions when B% = 80% (or $\underline{b}$ = (104.00, 200.00)). Point 3, $\underline{f}$ = (107.50, 212.50), is the set of solutions when B% = 85% (or $\underline{b}$ = (110.50, 212.50)). Points 4 and 5 represent sets of solutions when B% = 84% and B% = 83%, respectively. It is obvious that points 1,2, and 5 are dominated solutions while points 3 and 4 are nondominated solutions (they are on the AB line). Since point 4 is derived from a more precise B%, B% = 84%, than that of B% = 85%, we choose point 4, $\underline{f}$ = (108.00, 210.00), as MAG and 84% as MAR for our problem.

Step 1.2 Display ISGP II - Initial Table.

Initial Table	max $f_1(\underline{x})$	max $f_2(\underline{x})$
$\underline{f}^*$	130.00	250.00
MAG(84%)	108.00	210.00
$\underline{f}^-$	0.00	0.00
$\underline{w}$	1.00	1.00

The initial table is presented to the DM. We assume the DM does

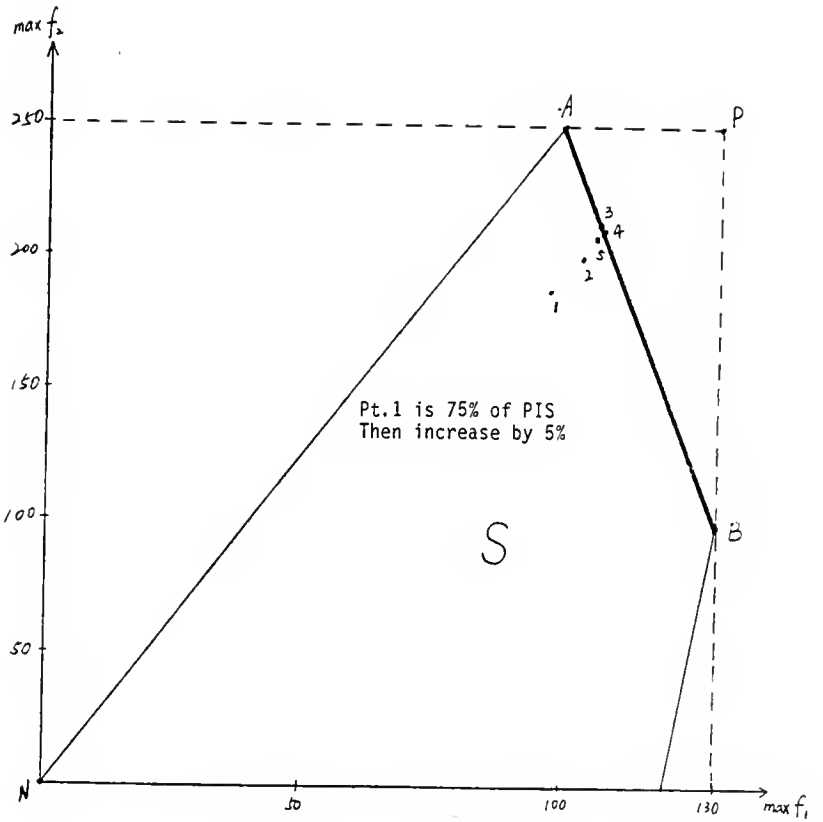

Fig. 4.1 The procurement of MAR and MAG for Hardee Co. production schedule problem.

not accept the MAG (84%) as the preferred solution.

Step 1.3 Set Initial Goals $\underline{b}$.

Assume the DM sets his/her initial goals $\underline{b}$ at 90% of $| f_1^* - f_1^- |$, i.e. $\underline{b}$ = (117.00, 225.00), referring to the ISGP II initial table.

Iteration No. 1.

Step 2.1 Solve for PS (Principal Solution).

The problem is to find $\underline{x}$ so as to

$$\min a_{ps} = \{d_1^- + d_2^-\}$$

$$\text{s.t.} \quad \underline{x} \in X$$

$$f_1(\underline{x}) + 130\ (d_1^- - d_1^+) = b_1 = 117$$

$$f_2(\underline{x}) + 250\ (d_2^- - d_2^+) = b_2 = 225$$

$$0 \leq d_1^-\ ,\ d_2^- \leq 1$$

This is a linear programming problem. The solution is given as

$$\underline{x} = (225.00,\ 50.00)$$

$$\underline{f}^o = (105.00,\ 225.00)$$

$$a_{ps} \neq 0$$

Step 2.2

As $a_{ps} \neq 0$ implies that the solution $\underline{f}^o$ is a nondominated solution and is accepted as the principal solution. Go to step 3.

Step 3 Solve for ASs (Auxiliary Solutions).

The first auxiliary solution is obtained by solving

$$\min \quad a^1 = \{ d_2^- \}$$

$$\text{s.t.} \quad \underline{x} \quad X$$

$$f_1(\underline{x}) \geq b_1 = 117$$

$$f_2(\underline{x}) + 250\,(d_2^- - d_2^+) = b_2 = 225$$

$$0 \leq d_2^-,\ d_2^+ \leq 1$$

This is a linear programming problem. The solution is given as

$$\underline{x} = (165.00,\ 170.00)$$

$$\underline{f}^1 = (117.00,\ 165.00)$$

$$a^1 \neq 0$$

Step 3.1

The answer $a^1 \neq 0$ implies $\underline{f}^1$ is a nondominated solution and is accepted as the first auxiliary solution. Go to step 3 for the second auxiliary solution set.

Step 3 Next, we obtain the second auxiliary solution by solving

$$\min \ a^2 = \{ d_1^- \}$$

$$\text{s.t.} \quad \underline{x} \in X$$

$$f_1(\underline{x}) + 130\,(d_1^- - d_1^+) = b_1 = 117$$

$$f_2(\underline{x}) \geq b_2 = 225$$

$$0 \leq d_1^-,\ d_1^+ \leq 1$$

This is a LP problem. The solution is given as

$\underline{x}$ = (225.00, 50.00)

$\underline{f}^2$ = (105.00, 225.00)

$a^2 \neq 0$

Step 3.1

Since $a^2 \neq 0$ we accept $\underline{f}^2$ as the second auxiliary solution.

Step 4 Form ISGP II - Tradeoff Table.

	max $f_1(\underline{x})$	max $f_2(\underline{x})$
f^*	130.00	250.00
MAG(84%)	108.00	210.00
$\underline{f}^-$	0.00	0.00
$\underline{w}$	1.00	1.00
Iteration 1		
Goal	117.00	225.00
PS	105.00	225.00
AS1	117.00	165.00
AS2	105.00	225.00

This tradeoff table is presented to the DM for evaluation.

Step 5 Evaluation phase.

We assume the DM is satisfied with none of the solutions. We proceed to Step 5.2 for setting another set of goals $\underline{b}$.

Step 5.2 Set a set of new goals $\underline{b}$.

Since none of the PS and ASs are satisfactory to the DM, a set of new goals $\underline{b}$ is set at (110.00, 215.00) by the DM.

Step 6 Calculation of $\hat{f}^-$ and $\hat{f}^+$.

The new goals, $\underline{b}$ = (110.00, 215.00) are compared with $\underline{f}^o$ = (105.00, 225.00) of the first iteration.

$$\hat{f}_1^+ = 110.00 - 105.00 = 5.00 \text{ because of } b_1 > f_1^o$$

$$\hat{f}_2^- = 225.00 - 215.00 = 10.00 \text{ because of } b_2 < f_2^o$$

Step 7 Consistency check on new $\underline{b}$.

There is one $\hat{f}^- > 0$ and one $\hat{f}^+ > 0$. That is, the new goals $\underline{b}$ is a set satisfying the consistent test and is accepted as the initial goals for the second iteration. Go back to Step 2.1 .

Iteration No.2

Step 2.1 Solve for PS (Principal Solution).

The problem of (4.6) is solved again with b_1 = 110.00 and b_2 = 215.00 . The solution is given as

$$\underline{x} = (215.00, 70.00)$$

$$\underline{f}^o = (107.00, 215.00)$$

$$a^{ps} \neq 0$$

Step 2.2 The answer $a^{ps} \neq 0$ implies that $\underline{f}^o$ is a set of nondominated solutions and is accepted as the principal solution.

Go to Step 3.

Step 3 Solve for ASs (Auxiliary Solutions).

Similar to that of iteration 1. We solve (4.7) with b_1 = 110.00 and b_2 = 215.00. The solution is given as

$\underline{x}$ = (200.00, 100.00)

$\underline{f}^1$ = (110.00, 200.00)

$a^1 \neq 0$

Step 3.1 The answer $a^1 \neq 0$ implies that $\underline{f}^1$ is a set of nondominated solutions and is accepted as the first auxiliary solution. Go to Step 3.

Step 3 This step solves for the second auxiliary solution.

Problem (4.8) was solved with b_1 = 110.00 and b_2 = 215.00 . The solution is given as

$\underline{x}$ = (215.00, 70.00)

$\underline{f}^2$ = (107.00, 215.00)

$a^2 \neq 0$

Step 3.1 The answer $a^2 \neq 0$ implies that $\underline{f}^2$ is a set of nondominated solutions and is accepted as the second auxiliary solution. Since both goals have been satisfied serially, we proceed to Step 4.

Step 4 Form the ISGP II - Tradeoff Table.

ISGP II - Tradeoff Table

	max $f_1(\underline{x})$	max $f_2(\underline{x})$
$\underline{f}^*$	130.00	250.00
MAG(84%)	108.00	210.00
$\underline{f}^-$	0.00	0.00
$\underline{w}$	1.00	1.00
Iteration 2		
Goal	110.00	215.00
PS	107.00	215.00
AS1	110.00	200.00
AS2	107.00	215.00

This tradeoff table is presented to the DM for evaluation.

Step 5 Evaluation phase.

We assume the DM is satisfied with the first auxiliary solution AS1 and selects it as the preferred solution. Go to Step 8.

Step 8 Stop.

The final solution is printed as

$\underline{x}$ = (200.00, 100.00)

$\underline{f}$ = (110.00, 200.00)

and the algorithm terminates here.

4.3.2 Nutrition Problem.

This problem is to determine the quantities of six foods that should be eaten to balance nutritional requirements so as to

(i) maximize carbohydrate intake,

(ii) minimize cholesterol intake, and

(iii) minimize cost.

Information on the six foods is given in the following table.

Table 4.2 Nutrition and cost information of six foods

	Milk (pint)	Beef (pound)	Eggs (dozen)	Bread (ounce)	Lettuce salad (ounce)	Orange juice (pint)	daily intake for adults
Vitamin A (i.u.)	720	107	7080	0	134	1000	5000
Food energy (calories)	344	1460	1040	75	17.4	240	2500
Cholesterol (unit)	10	20	120	0	0	0	
Protein (g)	18	151	78	2.5	0.2	4	63
Carbohydrate(g)	24	27	0	15	1.1	52	
Iron (mg)	0.2	10.1	13.2	0.75	0.15	1.2	12.5
Cost (S)	0.22	2.2	0.8	0.1	0.05	0.26	

The problem is summarized as :

$$\max f_1(\underline{x}) = 24x_1 + 27x_2 + 15x_4 + 1.1x_5 + 52x_6 \quad \text{(Carbohydrate intake)}$$

$$\min f_2(\underline{x}) = 10x_1 + 20x_2 + 120x_3 \quad \text{(Cholesterol intake)}$$

$$\min f_3(\underline{x}) = 0.22x_1 + 2.2x_2 + 0.8x_3 + 0.1x_4 + 0.05x_5 + 0.26x_6 \quad \text{(Cost)}$$

Subject to:

$720x_1 + 107x_2 + 7080x_3 + 134x_5 + 1000x_6 \geq 5000$ (Vitamin A,i.u.)

$0.2x_1 + 10.1x_2 + 13.2x_3 + 0.75x_4 + 0.15x_5 + 1.2x_6 \geq 12.5$ (Iron,mg)

$344x_1 + 1460x_2 + 1040x_3 + 75x_4 + 17.4x_5 + 240x_6 \geq 2500$ (Calories)

$18x_1 + 151x_2 + 78x_3 + 2.5x_4 + 0.2x_5 + 4x_6 \geq 63$ (Protein, g)

$x_1 \leq 6.0$ (Milk, pints)

$x_2 \leq 1.0$ (8eff,pounds)

$x_3 \leq 0.25$ (Eggs, dozen)

$x_4 \leq 10.0$ (8read, ounces)

$x_5 \leq 10.0$ (Lettuce & Salad,ounces)

$x_8 \leq 4.0$ (Orange juice, pints)

$x_i \geq 0,\ i = 1,\ldots,6$

$J = \{1\},\quad I = \{2, 3\}$

The problem is solved by the following steps.

Step 0 Obtain and Construct PIS & NIS Payoff Tables.

PIS-Payoff Table

	f_1	f_2	f_3	x_1	x_2	x_3	x_4	x_5	x_6
max f_1	540.00*	80.00	6.09	6.00	1.00	0.00	10.00	10.00	4.00
min f_2	380.39	8.44*	3.47	0.00	0.42	0.00	10.00	10.00	4.00
min f_3	281.87	67.83	2.26*	3.78	0.00	0.25	10.00	0.00	0.79

$\underline{f}^* = (540.00, 8.44, 2.26)$

NIS- Payoff Table

	f_1	f_2	f_3	x_1	x_2	x_3	x_4	x_5	x_6
max f_1	93.34^-	71.53	3.08	2.50	0.83	0.25	0.00	10.00	0.00
min f_2	321.00	110.00^-	4.75	6.00	1.00	0.25	10.00	0.00	0.00
min f_3	540.00	110.00	6.29^-	6.00	1.00	0.25	10.00	10.00	4.00

$$\underline{f}^- = (93.34,\ 110.00,\ 6.29)$$

Step 1.1 Obtain a set of MAG and MAR.

Set B% = 75% (initially)

The problem is to find a set of $\underline{x}$ so as to

$$\min a^g = (d_1^- + d_2^- + d_3^-)$$

$$\text{s.t. } \underline{x} \in X$$

$$f_1(\underline{x}) + (d_1^- - d_1^+)(f_1^* - f_1^-) = b_1$$

$$f_2(\underline{x}) - (d_2^- - d_2^+)(f_2^- - f_2^*) = b_2$$

$$f_3(\underline{x}) - (d_3^- - d_3^+)(f_3^- - f_3^*) = b_3$$

$$0 \le d_r^-,\ d_r^+ \le 1 \quad ,\ r = 1,\ 2,\ 3$$

where $b_1 = 540.00 - (1 - 75\%)(540.00 - 93.34) = 428.33$

$b_2 = 8.44 + (1 - 75\%)(110.00 - 8.44) = 33.83$

$b_3 = 2.26 + (1 - 75\%)(6.29 - 2.26) = 3.27$

This is a LP problem. The solution is given as

$\underline{x}$ = (2.8060, 0.2885, 0, 10, 0, 3.9077)

$\underline{f}^g$ = (428.33, 33.83, 3.27)

a^g = 0

Since a^g = 0, $\underline{f}^g$ = (428.33, 33.83, 3.27) is a set of dominated solutions which is not qualified to serve as the problem's MAG. Therefore, we increase the 8% by 5%, i.e. 8% = 80%, and repeat the calculation of (4.9) and (4.10) and obtain the following answers:

b_1 = 450.67, b_2 = 28.75, b_3 = 3.07

$\underline{x}$ = (2.8752, 0, 0, 10.00, 7.8691, 4.00)

$\underline{f}^g$ = (435.66, 28.75, 3.07)

a^g = 0.336

Since $a^g \neq 0$, $\underline{f}^g$ = (435.66, 28.75, 3.07) is a nondominated solution. The MAR, 8%, is in the range between 76% and 80%. To identify the precise B% value, the following calculation is conducted.

We decrease 8% by 1%, i.e. 8% = 79%, calculate (4.9) and (4.10), and obtain the following solutions:

b_1 = 446.20, b_2 = 29.77, b_3 = 3.11

$\underline{x}$ = (2.9768, 0, 0, 10.00, 8.2221, 4.00)

$\underline{f}^g$ = (438.49, 29.77, 3.11)

a^g = 0.0173

Since $a^g \neq 0$, we decrease 8% by 1% again, i.e. 8% = 78%, and solve (4.9) and (4.10). The solutions are :

b_1 = 441.74, b_2 = 30.78, b_3 = 3.15

$\underline{x}$ = (3.0780, 0, 0, 10.00, 8.6568, 4.00)

$\underline{f}^g$ = (441.39, 30.78, 3.15)

a^g = 0.00096

Since $a^g \neq 0$, we reduce the B% from 78% to 77%, and solve (4.9) and (4.10) again. The solutions are :

b_1 = 437.27, b_2 = 31.80, b_3 = 3.19

$\underline{x}$ = (2.9491, 0.1154, 0, 10.00, 4.8857, 4.00)

$\underline{f}^g$ = (437.27, 31.80, 3.19)

a^g = 0.00

Since $a^g = 0$, we conclude that the MAR is 78%, i.e. B% = 78%, and the MAG is (441.39, 30.78, 3.15).

Step 1.2 Display ISGP II - Initial Table.

	max f_1	min f_2	min f_3
$\underline{f}^*$	540.00	8.44	2.26
MAG(78%)	441.39	30.78	3.15
$\underline{f}^-$	93.34	110.00	6.29
$\underline{w}$	1.00	1.00	1.00

This initial table is presented to the DM. If the DM is satisfied with the MAR (78%), then take the MAG as the preferred solution and go to step 8, otherwise go to step 1.3 .

Iteration No.1:

Step 1.3 Set initial goals $\underline{b}$.

We assume that the DM set his/her initial $\underline{b}$ at 90% of $|f_1^* - f_1^-|$, i.e. $\underline{b}$ = (495.33, 18.60, 2.66), referring to initial table.

Step 2.1 Calculate Principal Solution.

The problem is to find a set of $\underline{x}$ so as to

$$\min a_{ps} = \{d_1^- + d_2^- + d_3^-\}$$

s.t. $\underline{x} \in X$

$$f_1(\underline{x}) + (d_1^- - d_1^+)(f_1^* - f_1^-) = b_1 = 495.33$$

$$f_2(\underline{x}) - (d_2^- - d_2^+)(f_2^- - f_2^*) = b_2 = 18.60$$

$$f_3(\underline{x}) - (d_3^- - d_3^+)(f_3^- - f_3^*) = b_3 = 2.66$$

$$0 \leq d_1^-, d_2^-, d_3^- \leq 1$$

This is a LP problem. The solution is :

$\underline{x}$ = (2.1433, 0, 0, 10.00, 3.0196, 4.00)

$\underline{f}^o$ = (412.77, 21.43, 2.663)

a_{ps} = 0.0225

The answer $a_{ps} \neq 0$ indicates that $\underline{f}^o$ is a set of nondominated solutions: we take $\underline{f}^o$ as the principal solution.

Step 3 Calculate the first Auxiliary Solution (AS1).

The problem is to find a set of $\underline{x}$ so as to

.

$$\min a^1 = \{ d_2^- + d_3^- \}$$

$$\text{s.t.} \quad \underline{x} \in X$$

$$f_1(\underline{x}) \geq b_1 = 495.33$$

$$f_2(\underline{x}) - (d_2^- - d_2^+)(f_2^- - f_2^*) = b_2 = 18.60$$

$$f_3(\underline{x}) - (d_3^- - d_3^+)(f_3^- - f_3^*) = b_3 = 2.66$$

$$0 \leq d_2^-, d_3^- \leq 1$$

This is a LP problem. The solution is

$\underline{x}$ = (5.7222, 0, 0, 10.00, 0, 4.00)

$\underline{f}^1$ = (495.33, 57.22, 3.30)

a^1 = 0.5381

Step 3.1

$a^1 \neq 0$ indicates that $\underline{f}^1$ is a set of nondominated solutions; we take $\underline{f}^1$ as the first auxiliary solution. Go to step 3.

Step 3 Calculate the second Auxiliary Solution (AS2).

The problem is to find $\underline{x}$ so as to

$$\text{Min } a^2 = \{ d_1^- + d_3^- \}$$

$$\text{s.t.} \quad \underline{x} \in X$$

$$f_1(\underline{x}) + (d_1^- - d_1^+)(f_1^* - f_1^-) = b_1 = 495.33$$

$$f_3(\underline{x}) - (d_3^- - d_3^+)(f_3^- - f_3^*) = b_3 = 2.66$$

$$f_2(\underline{x}) \leq b_2 = 18.60$$

$$0 \leq d_1^-,\ d_3^- \leq 1$$

This is a LP problem. The solution is :

$\underline{x}$ = (1.8596, 0, 0, 10.00, 8.6378, 4.00)

$\underline{f}^2$ = (412.13, 18.60, 2.88)

a^2 = 0.2404

Step 3.1

$a^2 \neq 0$ indicates that $\underline{f}^2$ is a set of nondominated solution; we take $\underline{f}^2$ as the second auxiliary solution. Go to step 3

Step 3 Calculate the third Auxiliary Solution (AS3).

The problem is to find a set of $\underline{x}$ so as to

$$\min a^3 = (d_1^- + d_2^-)$$

s.t. $\underline{x} \in X$

$$f_1(\underline{x}) + (d_1^- - d_1^+)(f_1^* - f_1^-) = b_1 = 495.33$$

$$f_2(\underline{x}) - (d_2^- - d_2^+)(f_2^- - f_2^*) = b_2 = 18.60$$

$$f_3(\underline{x}) \leq b_3 = 2.66$$

$$0 \leq d_1^-,\ d_2^- \leq 1$$

This is a LP problem. The solution is:

$\underline{x}$ = (2.1433, 0, 0, 10.00, 3.0296, 4.00)

$\underline{f}^3$ = (412.77, 21.43, 2.66)

a^3 = 0.2128

Step 3.1

$a^3 \neq 0$ indicates that $\underline{f}^3$ is a set of nondominated solutions; we take $\underline{f}^3$ as the third auxiliary solution. We have satisfied every goal serially. Go to step 4.

Step 4 Form ISGP II - Tradeoff Table.

ISGP II - Tradeoff Table

	max $f_1(\underline{x})$	min $f_2(\underline{x})$	min $f_3(\underline{x})$
$\underline{f}^*$	540.00	8.44	2.26
MAG(78%)	441.39	30.78	3.15
$\underline{f}^-$	93.34	110.00	6.29
$\underline{w}$	1.00	1.00	1.00
Iteration 1			
Goal	495.33	18.60	2.66
PS	412.77	21.43	2.66
AS1	495.33	57.22	3.30
AS2	412.13	18.60	2.88
AS3	412.77	21.43	2.66

This table is presented to the DM for evaluation.

Step 5 Evaluation phase.

Step 5.1 We assume the DM is not satisfied with any of the solution sets. Go to Step 5.2.

Step 5.2 Set new goals set, $\underline{b}$.

We assume that the DM realizes that unless all his/her previous goals are relaxed, no compromise solution will be obtained. A new set of goals is given at 80% of $| f_1^* - f_1^- |$, i.e. $\underline{b}$ = (450.67, 28.75, 3.07).

Step 6 Calculate The Deviations Of The New Goals From PS.

$b_1 > f_1^0$ implies $\hat{f}_1^+ = 37.90 = 450.67 - 412.77$

$b_2 > f_2^0$ implies $\hat{f}_2^- = 7.32 = 28.75 - 21.43$

$b_3 > f_3^0$ implies $\hat{f}_3^- = 0.41 = 3.07 - 2.66$

Step 7 Consistency check for new $\underline{b}$.

There are two $\hat{f}_1^- > 0$ when one $\hat{f}_1^+ > 0$ and that implies the new goal set is acceptable. Go to step 2.1 and start the second iteration.

Iteration No.2

After a new goal $\underline{b}$ is given, the procedure in the first iteration is repeated.

The ISGP II - Tradeoff Table is given as:

	max $f_1(\underline{x})$	min $f_2(\underline{x})$	min $f_3(\underline{x})$
$\underline{f}^*$	540.00	8.44	2.26
MAG(78%)	441.39	30.78	3.15
$\underline{f}^-$	93.34	110.00	6.29
$\underline{w}$	1.00	1.00	1.00
Iteration 2			
Goal	450.67	28.75	3.07
PS	435.66	28.75	3.07
AS1	450.67	36.58	3.07
AS2	435.66	28.75	3.07
AS3	435.66	28.75	3.07

The tradeoff table is presented to the DM for evaluation.

Evaluation:

The ISGP II - Tradeoff Table is presented to the DM. The DM finds that when b_2 and b_3 are satisfied, b_1 receives less attention than expected. Again, unless all goals are relaxed, there won't be any agreement on the possible solutions.

Set new goals:

The new goal is set at $\underline{b}$ =(442.00, 30.00, 3.10).

Calculate the deviations of the new goals to PS:

$b_1 > f_1^0$ implies $\hat{f}_1^+ = 6.36 = 442.00 - 435.66$

$b_2 > f_2^0$ implies $\hat{f}_2^- = 1.25 = 30.00 - 28.75$

$b_3 > f_3^0$ implies $\hat{f}_3^- = 0.03 = 3.10 - 3.07$

Consistency check:

There are two $\hat{f}_1^- > 0$ when one $\hat{f}_1^+ > 0$ and that implies the new goal set is acceptable. Go to step 2.1 and start the third iteration.

Iteration No.3

The procedure in iteration 1 is repeated. The ISGP II - Tradeoff Table is given as:

	max $f_1(\underline{x})$	min $f_2(\underline{x})$	min $f_3(\underline{x})$
$\underline{f}^*$	540.00	8.44	2.26
MAG(78%)	441.39	30.78	3.15
$\underline{f}^-$	93.34	110.00	6.29
$\underline{w}$	1.00	1.00	1.00
Iteration 3			
Goal	442.00	30.00	3.10
PS	438.00	30.00	3.10
AS1	442.00	31.67	3.10
AS2	438.00	30.00	3.10
AS3	438.00	30.00	3.10

The tradeoff table is presented to the DM for evaluation.

Evaluation:

The ISGP II - Tradeoff Table is presented to the DM. Assume the DM is satisfied with the PS, i.e., the final preferred solution is reached. Go to step 8.

Step 8

The print-out of the preferred solution is given as:

$\underline{x}$ = (3.00, 0, 0, 10.00, 8.00, 4.00)

$\underline{f}$ = (438.00, 30.00, 3.10).

And the ISGP II algorithm terminates.

4.4 Remarks:

1. The algorithm for ISGP II is an efficient procedure for solving MODM problems with multiple DMs, especially the incorporation of information of $\underline{f}^*$, $\underline{f}^-$, MAG, MAR, the principal and auxiliary solutions. ISGP II can act as a source of information for the DMs and also help in finding consensus agreements among DMs within a few iterations. The major advantage of ISGP II is that any single objective optimization technique can be used to obtain nondominated solutions. However, ISGP II can only handle linear models in which a convex set is guaranteed in forming a feasible region. If the feasible region is a non-convex set, caution should be paid in using ISGP II. (See Fig. 4.2 and Fig. 4.3 for graphical demonstration of convex and non-convex set.)

2. If the overachievement variables, $\underline{d}^+$, are not included in expressions of ISGP II, converting the objectives to constraints, the formulation is given as

$$f_j(\underline{x}) + d_j^- \left(f_j^* - f_j^- \right) = b_j, \quad j \in J \tag{4.12}$$

$$f_i(\underline{x}) - d_i^- \left(f_i^- - f_i^* \right) = b_i, \quad i \in I$$

When using this set of formulations in the nutrition problem, we obtained an infeasible solution. For example, we intentionally selected a set of $\underline{b}$ = (94, 110, 6.20) and formed a LP model as follows:

$$\min \left(\sum_{r=1}^{3} d_r^- \right)$$

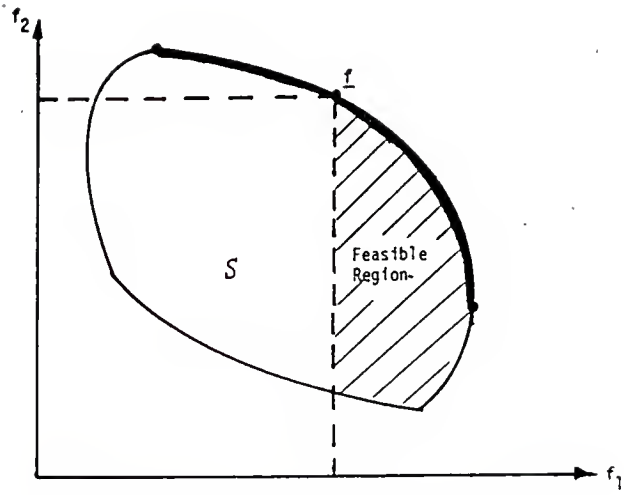

Fig. 4.2 Convex with two objectives in nonlinear problem.

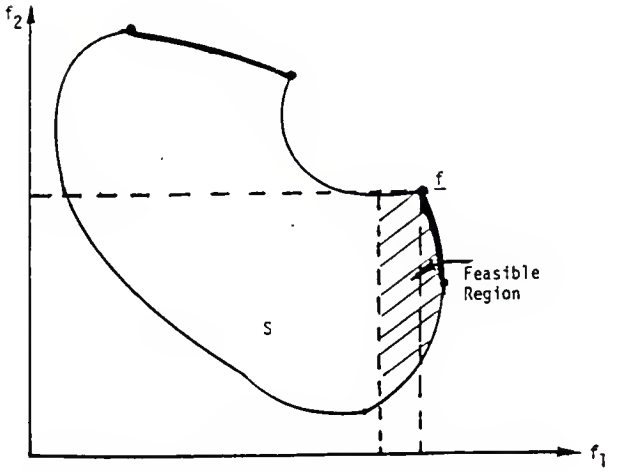

Fig. 4.3 Nonconvex with two objectibes in nonlinear problem.

s.t. $\underline{x} \in X$

$$f_1(\underline{x}) + 446.66\ d_1^- = 94$$

$$f_2(\underline{x}) - 101.56\ d_2^- = 110$$

$$f_3(\underline{x}) - 4.03\ d_3^- = 6.20$$

$$0 \leq d_r^- \leq 1, \quad r = 1, 2, 3$$

This LP problem gives an infeasible solution. In order to amend that pitfall, $\underline{d}^+$ should be included into expression (4.12) and (4.13) should be formulated as follows:

$$f_j(\underline{x}) + (d_j^- - d_j^+) (f_j^* - f_j^-) = b_j, \quad j \in J \qquad (4.13)$$

$$f_i(\underline{x}) - (d_i^- - d_i^+) (f_i^- - f_i^*) = b_i, \quad i \in I$$

The test results for (4.12) and (4.13) are given in Table 4.3. (Notice that the achievement function for both formulations is $\min \{ \sum_{r=1}^{3} d_r^- \}$)

Table 4.3 Test results for nutrition problem

	expression (4.12)	expression (4.13)
$\underline{b}$	$\underline{f}$	$\underline{f}$
(94, 110, 6.20)	infeasible	(396.52, 17.99, 2.83)
(300, 100, 5.00)	(300, 108.89, 5.00)	(396.52, 17.99, 2.83)
(400, 20, 2.50)	(400, 24.05, 2.55)	(413.11, 22.97, 2.55)
(500, 100, 3.00)	(500, 100, 4.56)	(462.72, 43.64, 3.00)
(539, 8.50, 2.30)	(413.11, 22.97, 2.55)	(413.11, 22.97, 2.55)

It is obvious that the results of (4.13) are at least equal to or better than those of (4.12). Cases such as $\underline{b}$ = (94, 110, 6.20), $\underline{b}$= (300, 100, 5.00), and $\underline{b}$ = (400, 20, 2.50) confirmed that expression (4.13) is indeed superior than expression (4.12).

Chapter 5. CONCLUSIONS

Multiple Criteria Decision Making problems are all around us. Problems in private and public sectors occur in such a way that multiple, conflicting, noncommensurable decision criteria are not uncommon. To make decisions based on intuitive thinking is no longer applicable, hence, mathematical programming techniques are introduced to help the DM make rational decisions. The methods that incorporate mathematical programming techniques are also known as MCDM solution methods. These methods are broadly classified as: (1) MADM: methods that deal with a finite (usually small) number of predetermined alternatives. (2) MODM: methods that deal with an infinite number of alternatives, and usually requires design and search efforts.

The methods discussed in earlier chapters are TOPSIS-MADM, TOPSIS-MODM, and ISGP II. We will discuss them briefly in this section. (1) TOPSIS-MADM: the idea of the method is that a preferred alternative should be the one that is closest to the positive-ideal point and farthest from the negative-ideal point. The terms "closeness" and "farthest" are measured by distances. The decision matrix must be normalized before any comparison is made because the attributes may have different measurement units. The parameter, q, for the normalization equation can be 1, 2, or ∞ . Similarly, the parameter, p, for distance measurement function can also be 1, 2, or ∞. Different q and p combinations may result in conflicting solutions. To resolve the conflict, the

relative reliability of a distance measurement, $\underline{C}$, is introduced to harmonize the final solutions. (2) TOPSIS-MODM: the concept of this method is inheritated from TOPSIS-MADM, i.e. closest to the positive-ideal and farthest to the negative-ideal. The mathematical structure of this method is adapted from the global criteria method, i.e. the distance-measuring function. Although the alternative that is closest to the positive-ideal as well as farthest from the negative-ideal rarely exists, we can manage that by introducing the importance index, $\underline{H} = (H_p, H_n)$, of distances to PIS and NIS respectively. And Similar to that of TOPSIS-MADM, the parameter, p, in distance function can be 1, 2, and oo. Each p value results in a different set of nondominated solutions. The OM may select one or none of the three solutions. (3) ISGP II: ISGP II combines attractive features of both Goal Programming and Interactive methods. The DM is actively involved in the decision making process while the necessary information from the OM is simplified. The DM gains insight into the problem due to the iterative nature of the algorithm. The cases with multiple decision makers are handled by introducing the Maximum Achievable Goals (MAG) and the Maximum Achievable Rate (MAR). So far, linear problems have been successfully solved by ISGP II.

The development of MCOM methods is more and more oriented twoard practical uses. The computerized algorithms that allow man-machine "conversation" have also gained a lot of attention. MCOM methods are becoming more and more powerful in their effort to aid human beings in making rational decisions.

REFERENCES

1. Benayoun, R., J. de Montgolfier, J. Tergny, and O. Larichev, "Linear Programming with Multiple Objective Functions: Step Method (STEM)", Mathematical Programming, Vol. 1, No. 3, pp. 366-375, 1971.

2. Benson, R.G., "Interactive Multiple Criteria Optimization Using Satisfactory Goals," Ph.D. Thesis, University of Iowa, 1975.

3. Charnes, A., W.W. Cooper, Management Models and Industrial Applications of Linear Programming, Volume I, Wiley New York, 1961.

4. Cohon, J.L., Multiobjective Programming and Planning, Academic Press, New York, 1978.

5. Cohon, J.L., and M. Zeleny (eds.), Multiple Criteria Decision Making, University of South Carolina Press, Columbia, South Carolina, 1973.

6. Dyer, J.S.,"Interactive Goal Programming", Management Science, Vol. 19, No. 1, pp. 62-70, 1972.

7. Fichefet, J., "GPSTEM: An Interactive Multiobjective Optimization Method", in A. Prekopa (ed.), Progress in Operations Research, Vol. 1, pp. 317-332, North Holland, Amsterdam, 1976.

8. Goicoechea, A., D.R. Hansen, and L. Duckstein, Multiobjective Decision Analysis With Engineering And Business Applications, Wiley, New York, 1982.

9. Haimes, Y.Y., W.A. Hall and H.T. Freedman, Multiobjective Optimization in Water Resources Systems, The Surrogate Worth Trade-Off Method, Elsivier Scientific, New York, 1975.

10. Hwang, C.L., A.S.M. Masud, In Collaboration with S.R. Paidy and K. Yoon, Multiple Objective Decision Making-Methods and Applications: A Stat-of-the-Art Survey, Springer-Verlag, Berlin/Heidelberg/New York,1979.

11. Hwang, C.L., K. Yoon, Multiple Attribute Decision Making Methods and Applications: A State-of-the-Art Survey, Springer-Verlag, Berlin/Heidelberg/New York, 1981.

12. Hwang, C.L., A.S.M. Masud,"Interactive Sequential Goal Programming," Operations Research, Vol. 32, pp.391-400, 1981.

13. Ignizio, J.P., Goal Programming and Extensions, Lexington Books, Massachusets, 1976.

14. Ijiri, Y., Management Goals And Accounting For Control, North-Holland, North-Holland, Amsterdam, 1965.

15. Lee, S.M., Goal Programming for Decision Analysis, Auerbach Publishers, Philadelphia, Pennsylvania, 1972.

16. Monarchi, D.E.,"Interactive Algorithm for Multiple Objective Decision Making", Technical Reports on Hydrology and Water Resources, (No. 6), University of Arizona, 1972.

17. Monarchi, D.E., C.C. Kisiel, and L. Duckstein, "Interactive Multi-Objective Programming in Water Resources: A Case Study", Water Resources Research, Vol. 9, No. 4, pp. 837-850, 1973.

18. Yoon, K., "A Reconciliation Among Discrete Compromise Solutions," to be published, 1986.

19. Yu, P.L., Multiple-Criteria Decision Making: Concepts, Techniques, and Extensions, Plenum Press, New York, 1985.

20. Zeleny, M., Linear Multiobjective Programming, Springer-Verlag, New York, 1974.

21. Zeleny, M. (ed.), Multiple Criteria Decision Making: Kyoto 1975, Springer-Verlag, New York, 1976.

22. Zeleny, M., Multiple Criteria Decision Making, McGraw-Hill, New York, 1982.

23. Zeleny, M. (ed.), MCDM: Past Decade And Future Trends: A Source Book of Multiple Criteria Decision Making, Jai Press, Greenwich, Connecticut, 1984.

24. Fleischer, G. A., Engineering Economy, PWS Engineering, Boston, Mass., 1984

Appendix A.

This BASIC program is used to generate the mean value of d_p, $E(d_p)$, and the relative reliability rate C_p. The program is from Yoon [1986].

```
      PROGRAM AA
      DIMENSION X (50)
      REAL NSAM, NSBM, NSCM

      WRITE (5,*)'HOW MANY VARIABLE(FIRST & LAST)'
      READ (5,90)LS,L LVC
      WRITE (5,*)'HOW MANY RANDOM GENERATION'
      READ (5,91)PRAN

      WRITE(5,200)
      WRITE(5,210)LS,LVC
      WRITE(5,220)PRAN
      WRITE(3,200
      WRITE(3,210)LS,LVC
      WRITE(3,220)PRAN
      WRITE(3,100)
      WRITE(5,100)
      WRITE(3,110)
      WRITE(5,110)

      DO 10 LV = LS,LVC,1
      AK=0.0
      BK=0.0
```

```
      CK=0.0
      SA=0.0
      SP=0.0
      SC=0.0
      SAM=0.0
      SBM=0.0
      SSM=0.0
      ZK=0.0
      QK=0.0
      PM=0.0
      NSAM=0.0
      NSBM=0.0
      NSCM=0.0

      DO 15 ZK-1.0,PRAN,1.0
      QK=QK+1.0
      AK=0.0
      BK=0.0
      CK=0.0
      PB=0.0
      DO 20 M=1,LV,1
      X(M)=RAN(I,J)
20    CONTINUE
      M=1
      P=X(M)
      DO 30 M=2,LV,1
      IF(P.LT.X(M)) GO TO 35
```

```
          P=P
          30 ID 30
 35       P=X (M)
 30       CONTINUE
          DO 40 M=1,LV.1
          AK=AK+X(M)
          PB=PB+X(M)**2
 40       CONTINUE
DUAO: [USERIE.KY175]AA.FOR:59
          BK=PB**.5
          CK=P
          SA=SA+AK
          SB=SB+BK
          SC=SC+CK
 15       CONTINUE
          SAM=SA/QK
          SBM=SB/QK
          SCM=SC/QK
          PM=SAM+SBM+SCM
          NSAM=SAM/PM
          NSBM=SBM/PM
          NSCM=SCM/PM

          WRITE(3,400)LV,SA,SB,SC,SAM,SBM,SCM,NSAM,NSBM,NSCM
          WRITE(5,400)LV,SA,SB,SC,SAM,SBM,SCM,NSAM,NSBM,NSCM
 10       CONTINUE

          WRITE(3,110)
```

```
      WRITE(5,110)
91    FORMAT(F12.1)
90    FORMAT(2IB)
100   FORMAT(///5X,'# OF VAL',9X,'SA',14X,'SB',14X,'SC',11X,
     & 'SAM',7X,'SBM',7X,'SCM,7X,'NSAM',6X,'NSBM',6X,'NSCM')
110   FORMAT(3X,120('-'))
200   FORMAT(//12X,'* SIMULATION CONDITION *')
210   FORMAT(8X,'NUMBER OF VALIABLES - FROM',I3,'TO', I3)
220   FORMAT(8XM'NUMBER OF RANDOM GENERATION TIME -', F8.0)
400   FORMAT(/5X,I3,GX,3F15.4,3X,GF10.4)
      STOP
      SND
```

Appendix B

THE METHOD OF GLOBAL CRITERION

The method of global criterion is to obtain an optimal vector that minimizes some global criterion; for example, the sum of the squares of the relative deviations of the criteria (objective functions) from the ideal points.

The method is used to obtain a solution for the MODM problem formulated below:

$$\max/\min\ [f_1(\underline{x}),\ f_2(\underline{x}),\ f_3(\underline{x}),\ \ldots,\ f_k(\underline{x})] \tag{3.1}$$

$$\text{s.t.}\quad \underline{x} \in X \quad \text{where } X = \{\underline{x} \mid g_l(\underline{x})\ \{\geq = \leq\}\ 0,\ l=1,2,\ldots,m\}$$

where $f_j(\underline{x})$: benefit objective for maximization, $j \in J$

$f_i(\underline{x})$: cost objective for minimization, $i \in I$

The method of global criterion for the vector problem of (3.1) solves the problem given by (3.2):

$$\min_{\{\underline{x}\}} F_p = \sum_{j\in J} w_j \left[\frac{f_j^* - f_j(\underline{x})}{f_j^* - \bar{f}_j} \right]^p + \sum_{i\in I} w_i \left[\frac{f_i(\underline{x}) - f_i^*}{\bar{f}_i - f_i^*} \right]^p \tag{3.2}$$

$$\text{s.t.}\quad \underline{x} \in X,$$

$$p = 1,2,\infty$$

where w_l, $l = 1,2,\ldots,k$ are the relative importance (weights) of objectives. Boychuk and Ovchinnikov [1973] have suggested $p = 1$, and Salukvadze [1972,1974] has suggested $p = 2$ for the global criterion. Although global criteria p can be any value between 1 and ∞, distances with $p = 1$, 2, and ∞ are especially operationally important. Distance d_1 (the Manhattan distance, or city block distance) and d_2 (the Euclidean distance) are the longest and shortest distances in the geometric sense; distance d_∞ (the Tchebycheff distance) is the shortest distance in the numerical sense. The 'best'

soiution chosen for the MODM will differ greatiy according to the criteria chosen. These criteria locate points which are geometricaiiy different from one another. The probiem with this approach is to determine which p would resuit in a soiution that is the most satisfactory soiution to the Decision Maker (DM). It is aiso possibie that whatever p is chosen, (3.2) might give a soiution which has a particular objective value, $f_k(\underline{x})$, that is totally unacceptabie to the DM.

Case 1. p = 1; $\underline{f}(\underline{x})$ and $\underline{g}(\underline{x})$ are iinear functions.

When all the objectives, $f_k(\underline{x})$, $k = 1, 2, \ldots, k$, and all the constraint functions, $g_i(\underline{x})$, $i = 1, \ldots, m$, are iinear functions, and p = 1, then (3.2) becomes a Linear Programming (LP) probiem. The simpiex method for LP can soive the probiem.

Case 2. p = 2; $\underline{f}(\underline{x})$ and $\underline{g}(\underline{x})$ are iinear functions.

When both $\underline{f}(\underline{x})$ and $\underline{g}(\underline{x})$ are linear functions and p = 2, (3.2) becomes a convex programming problem; more precisely, a quadratic programming probiem.

As is well known, in a convex programming probiem, a iocal minimum is absoiute (or giobai). The convex programming probiem can be soived by using any method of finding a locai extreme, inciuding the gradient method, quadratic programming, or nonlinear programming methods.

The case where p = 3, 4, ...etc are normaliy quite simiiar to the case where p = 2. We wiii now consider the case where $p = \infty$ and show how it can be reduced to an LP probiem.

Case 3. $p = \infty$; $\underline{f}(\underline{x})$ and $\underline{g}(\underline{x})$ are linear functions.

$$\min \quad F_\infty = \sum_{j \in J} w_j \left[\frac{f_j^* - f_j(\underline{x})}{f_j^* - f_j^-} \right]^\infty + \sum_{i \in I} w_i \left[\frac{f_i(\underline{x}) - f_i^*}{f_i^- - f_i^*} \right]^\infty$$

$$\text{s.t.} \quad \underline{x} \in X \tag{3.3}$$

(3.3) can be transformed to (3.4) for solution of the LP problem (see Note (4) for this transformation):

$$\min \quad d_\infty \qquad \text{(minimizing deviation)}$$

$$\text{s.t.} \quad \underline{x} \in X$$

$$w_j \left[\frac{f_j^* - f_j(\underline{x})}{f_j^* - f_j^-} \right] \le d_\infty \qquad (\forall\, j \in J) \tag{3.4}$$

$$w_i \left[\frac{f_i(\underline{x}) - f_i^*}{f_i^- - f_i^*} \right] \le d_\infty \qquad (\forall\, i \in I)$$

This is a MINIMAX approach problem. Among all d_∞ terms in the constraints we are to minimize the maximum term of d_∞.

Case 4. $\underline{f}(\underline{x})$ and $\underline{g}(\underline{x})$ are nonlinear functions.

If any one or all of the objectives, $f_i(\underline{x})$, $i = 1, 2, \ldots, k$, and the constraints, $g_i(\underline{x})$, $i = 1, 2, \ldots, m$, are nonlinear functions, (3.2) is a nonlinear programming problem (for both $p = 1$ and $p = 2$). For $p = \infty$ and one or more than one of $f_i(\underline{x})$ and/or $g_i(\underline{x})$ are nonlinear functions, (3.4) format solves the problem, but it is a nonlinear programming problem. A variety of nonlinear programming methods (e.g. the sequential unconstrained minimization technique (SUMT) [Hwang, et al., 1975], or the generalized reduced gradient method (GRG) [Abadie & Carpenter, 1969; Hwang et al., 1972]) can be used for solving the problem. A new approach which is called iterative nonlinear goal

programming is an effective method for solving this problem (Hwang & Masud, 1979; see Sec. 2.2.2.b).

NUMERICAL EXAMPLE 1. (Production Scheduling of the Hardee Company)

The Hardee company makes two kinds of dolls. Doll A is of higher quality and Doll B is of lower quality. The respective profits are \$0.40 and \$0.30 per doll. Each Doll A requires twice as much labor as a Doll B, and if all dolls were of type B, the company could make 500 per day. The supply of material is sufficient for only 400 dolls per day (both A and B combined). The problem assumes that all the type A and type B dolls the factory can make could be sold, and that the best customer of the company wishes to have as many as possible of type A doll. The manager realizes that two objectives: (1) the maximization of profit, and (2) the maximization of production of Doll A, should be considered in scheduling their production.

This VMP can be formulated as follows:

$$\max f_1(\underline{x}) = 0.4x_1 + 0.3x_2$$

$$\max f_2(\underline{x}) = x_1$$

subject to

$$\left.\begin{aligned} g_1(\underline{x}) = x_1 + x_2 \le 400 \\ g_2(\underline{x}) = 2x_1 + x_2 \le 500 \\ x_1, x_2 \ge 0 \end{aligned}\right\} \quad \underline{x} \in X$$

where x_1 and x_2 are the number of Doll A and Doll B produced respectively.

The steps of the method of global criterion are as follows:

Step 0. Construct the Positive-Ideal Solution (PIS) and Negative-Ideal Solution (NIS) payoff tables.

It involves solving the objectives as individual LP problems:

0.1 (PIS) 1) max $f_1(\underline{x}) = 0.4x_1 + 0.3x_2$

s.t. $\underline{x} \in X$

The solution is: $x_1 = 100$, $x_2 = 300$; $f_1(x^*) = 130$

Point B is the ideal solution for $f_1(\underline{x})$ (See Figs. 3.1 and 3.2).

Similarly for the second objective:

2) max $f_2(\underline{x}) = x_1$

s.t. $\underline{x} \in X$

The solution is: $x_1 = 250$, $x_2 = 0$; $f_2(\underline{x}^*) = 250$

Point C is the ideal solution for $f_2(\underline{x})$.

The PIS payoff table is therefore:

	f_1	f_2	x_1	x_2
f_1	130*	100	100	300
f_2	100	250*	250	0

Note that in this payoff table, row 1 gives the solution vector $\underline{x}^* = (100, 300)$ which maximizes the objective $f_1(\underline{x}^*) = 130$; $f_2(100,300) = 100$ is the value taken on by the objective f_2 when f_1 reaches its maximum. Similarly, row 2 gives the solution vector $\underline{x}^* = (250, 0)$ which maximizes $f_2(\underline{x}^*) = 250$; $f_1(250, 0) = 100$ is the value taken on by f_1 when f_2 reaches its maximum.

The PIS is point P, $(f_1^*, f_2^*) = (130,250)$ shown in Figure 3.2. This is a utopia solution, which is not feasible.

0-2. (NIS) 1) min $f_1(\underline{x}) = .4x_1 + .3x_2$

s.t. $\underline{x} \in X$

The solution is: $x_1 = 0, \quad x_2 = 0; \quad f_1(\underline{x}^-) = 0$

2) $\min \; f_2(\underline{x}) = x_1$

$\text{s.t.} \quad \underline{x} \in X$

The solution is: $x_1 = 0, \; 0 \le x_2 \le 400; \quad f_2(\underline{x}^-) = 0$

The NIS payoff table is therefore:

	f_1	f_2	x_1	x_2
f_1	0−	0	0	0
f_2	120	0−	0	$0 \le x_2 \le 400$

The NIS is point N, $(\bar{f}_1, \bar{f}_2) = (0,0)$, shown in Figure 3.2.

Step 1. Obtain the preferred solution.

A preferred solution is a nondominated solution which is a point on the segment of straight line BC. (In Figs. 3.1 and 3.2 all points on the segment of straight line BC are nondominated solutions). Assume equal weights for the two objectives ($w_1 = w_2 = 0.5$).

Case 1. p = 1

The preferred solution for p = 1 in (3.2) is to find $\underline{x} = (x_1, x_2)$ so as to

$$\min F_{p=1} = 0.5^* \left[\frac{130 - (0.4x_1 + 0.3x_2)}{130 - 0} \right] + 0.5^* \left[\frac{250 - x_1}{250 - 0} \right]$$

$$= 1 - 0.00354x_1 - 0.00115x_2$$

$\text{s.t.} \quad \underline{x} \in X$

This is a linear programming problem. The solution is given by

$\underline{x} = [250, 0]$

$\underline{f} = [100, 250]$

which is point C in Figs. 3.1 and 3.2.

Case 2. p = 2

The preferred solution for p = 2 in (3.2) is to find $\underline{x} = (x_1, x_2)$ so as to

$$\min F_{p=2} = 0.5 * \left[\frac{130 - (0.4x_1 + 0.3x_2)}{130} \right]^2 + 0.5 * \left[\frac{250 - x_1}{250} \right]^2$$

$$\text{s.t.} \quad \underline{x} \in X$$

This is a noniinear programming probiem. The sequentiai unconstrained minimization technique (SUMT) [Hwang. et ai, 1975] is used for solving the problem. The soiution is

$$\underline{x} = [230.70, 38.60]$$

$$\underline{f} = [103.90, 230.70]$$

which is point D in Figs. 3.1 and 3.2.

Case 3. p = ∞

The preferred soiution for p = ∞ in (3.2) is to find $\underline{x}$ so as to

$$\min \quad d_\infty$$

$$\text{s.t.} \quad \underline{x} \in X$$

$$0.5 \left[\frac{130 - (.4x_1 + .3x_2)}{130 - 0} \right] \leq d_\infty$$

$$0.5 \left[\frac{250 - x_1}{250 - 0} \right] \leq d_\infty$$

This is a LP probiem. We can use the LP simpiex method to soive it. The solution is:

$$\underline{x} = [210, 84]$$

$$\underline{f} = [109.2, 210]$$

which is point E in Figs 3.1 and 3.2.

As shown in Fig. 3.1 for p = 1, the function to be minimized, $F_{p=1}$, is a linear function; therefore the preferred solution is determined by the slope of line BC and that of $F_{p=1}$ (i.e., min $F_{p=1} = 1 - 0.00354\ x_1 - 0.00115\ x_2$ is equivalent to max $(-F_{p=1}) = 0.00354\ x_1 + 0.00115\ x_2 - 1$.) Since the slope of $F_{p=1}$ is less than that of line BC (the line representing the nondominated solutions), point C is the preferred solution. For p = 2, the function to be minimized, $F_{p=2}$, is a nonlinear (quadratic) function, and the minimum point for $F_{p=2}$ is point D in Fig. 3.1, which is the point where the $F_{p=2}$ curve touches the segment of line BC. For p = ∞, the function to be minimized, $F_{p=\infty}$, cannot be drawn on Fig. 3.1 but can be represented in Fig. 3.2 as point E. Note that point E is the point that the largest square drawn between PIS and BC touches line BC in a normalized objective function space representation.

The DM's preferred solution for this example should be one of the nondominated solutions (represented by points on the segment of straight line between B and C in Figs. 3.1 and 3.2).

The solutions for $F_{p=1}$, point C, and for $F_{p=2}$, point D, and $F_{p=\infty}$, point E, derived by this method may not satisfy the DM at all.

NUMERICAL EXAMPLE 2. (The Nutrition Problem)

The nutrition problem is to find the quantities of certain foods that should be eaten to meet certain nutritional requirements at a minimum cost. Information on six foods is given in Table 3.1.

The problem is to find the quantities of these foods that satisfy: (i) the minimum daily requirement of vitamin A and iron; and the balanced daily requirement of food energy and protein; (ii) the minimum cholesterol daily intake; (iii) the maximum carbohydrate daily intake; and (iv) the minimum cost.

The maximum daily intakes of the foods are: 6 pints of milk, 1 pound of beef, 1/4 dozen eggs, 10 ounces of bread, 10 ounces of lettuce and salad, and 4 pints of orange juice.

Let the daily diet requirements of milk (in pints), beef (in pounds), eggs (in dozens), bread (in ounces), lettuce and salad (in ounces), and orange juice (in pints) be x_1, x_2, x_3, x_4, x_5, and x_6, respectively. Then, the MODM problem is to find $\underline{x} = (x_1, x_2, \ldots, x_6)$ so as to:

$$\max \quad f_1(\underline{x}) = 24x_1 + 27x_2 + 15x_4 + 1.1x_5 + 52x_6 \quad \text{(carbohydrate intake)}$$

$$\min \quad f_2(\underline{x}) = 10x_1 + 20x_2 + 120x_3 \quad \text{(cholesterol intake)}$$

$$\min \quad f_3(\underline{x}) = 0.225\,x_1 + 2.2x_2 + 0.8x_3 + 0.1x_4 + 0.05x_5 + 0.26x_6 \quad \text{(cost)}$$

$$\text{s.t.} \quad g_1(\underline{x}) = 720x_1 + 107x_2 + 7080x_3 + 134x_5 + 1000x_6 \geq 5000$$

$$g_2(\underline{x}) = 0.2x_1 + 10.1x_2 + 13.2x_3 + 0.75x_4 + 0.15x_5 + 1.2x_6 \geq 12.5$$

$$g_3(\underline{x}) = 334x_1 + 1460x_2 + 1040x_3 + 75x_4 + 17.4x_5 + 240x_8 \geq 2500$$

$$g_4(\underline{x}) = 18x_1 + 151x_2 + 78x_3 + 2.5x_4 + 0.2x_5 + 4x_6 \geq 63$$

$$g_5(\underline{x}) = x_1 \leq 8.0$$

$$g_8(\underline{x}) = x_2 \leq 1.0$$

$$g_7(\underline{x}) = x_3 \leq 0.25$$

$$g_8(\underline{x}) = x_4 \leq 10.0$$

$$g_9(\underline{x}) = x_5 \leq 10.0$$

$$g_{10}(\underline{x}) = x_6 \leq 4.0$$

$$x_i \geq 0, \quad i=1,2,\ldots,6$$

Let the constraints be denoted by $\underline{x} \in X$.

Table 3.1 Nutrition and cost information of six foods

	Milk (pint)	Beef (pound)	Eggs (dozen)	Bread (ounce)	Lettuce salad (ounce)	Orange juice (pint)	Recommended Daily Allowance for adults
Vitamin A (I.U.)	720	107	7080	0	134	1000	5000
Food energy (calories)	344	1460	1040	75	17.4	240	2500
Cholesterol (unit)	10	20	120	0	0	0	
Protein (g)	18	151	78	2.5	0.2	4	63
Carbohydrate (g)	24	27	0	15	1.1	52	
Iron (mg)	0.2	10.1	13.2	0.75	0.15	1.2	12.5
Cost ($)	0.225	2.2	0.8	0.1	0.05	0.26	

Step 0 Obtain the ideal solutions and construct payoff tables.

Using the same procedure as in Example 1, we obtain all the PIS's and NIS's. The payoff tables are given as follows:

Positive Ideal Payoff Table

	MAX Carb	MIN Chol	MIN Cost	Milk	Beef	Eggs	Bread	Lettuce	OJ
MAX Carb	540.00*	80.00	6.09	6.00	1.00	0.00	10.00	10.00	4.00
MIN Chol	380.39	8.44*	3.47	0.00	0.42	0.00	10.00	10.00	4.00
MIN Cost	281.87	87.83	2.26*	3.78	0.00	0.25	10.00	0.00	0.79

Negative Ideal Payoff Table

	MAX Carb	MIN Chol	MIN Cost	Milk	Beef	Eggs	Bread	Lettuce	OJ
MAX Carb	93.34⁻	71.53	3.08	2.50	0.83	0.25	0.00	10.00	0.00
MIN Chol	321.00	110.00⁻	4.75	6.00	1.00	0.25	10.00	10.00	0.00
MIN Cost	540.00	110.00	6.29⁻	6.00	1.00	0.25	10.00	10.00	4.00

Step 1 Obtain the preferred solution. (For $w_1 = w_2 = w_3 = 0.33$)

Case 1 p = 1:

$$\min \quad F_{p=1} = 0.33 \left[\frac{540 - (24x_1 + 27x_2 + 0x_3 + 15x_4 + 1.1x_5 + 52x_6)}{540 - 93.34} \right]$$

$$+ 0.33 \left[\frac{(10x_1 + 20x_2 + 120x_3) - 8.44}{110 - 6.44} \right]$$

$$+ 0.33 \left[\frac{(.225x_1 + 2.2x_2 + .8x_3 + .1x_4 + .05x_5 + .26x_6) - 2.26}{6.29 - 2.26} \right]$$

$$= .0993x_1 + .6824x_2 + 1.3801x_3 - .0088x_4 + .0099x_5$$

$$- .0519x_6 + .5651$$

s.t. $\underline{x} \in X$

This is a LP problem. The solution is

$$\underline{x} = [3.76,\ 0.01,\ 0.13,\ 10.00,\ 10.00,\ 0]$$

$$\underline{f} = [253.89,\ 55.51,\ 2.65]$$

Case 2 p=2:

$$\min F_{p=2} = 0.33 \left[\frac{540 - (24x_1 + 27x_2 + 0x_3 + 15x_4 + 1.1x_5 + 52x_6)}{540 - 93.34} \right]^2$$

$$+ 0.33 \left[\frac{(10x_1 + 20x_2 + 120x_3) - 6.44}{110 - 6.44} \right]^2$$

$$+ 0.33 \left[\frac{(.225x_1 + 2.2x_2 + .6x_3 + .1x_4 + .05x_5 + .26x_6) - 2.26}{6.29 - 2.26} \right]^2$$

s.t. $\underline{x} \in X$

This is a nonlinear programming problem. The Sequential Unconstrained Minimization Technique (SUMT) is used to solve this problem. The solution is

$$\underline{x} = [2.48,\ 0.13,\ 0.00,\ 9.94,\ 1.06,\ 4.00]$$

$$\underline{f} = [420.93,\ 27.75,\ 2.92]$$

Case 3. $p = \infty$

$$\min \quad d_\infty$$

$$\text{s.t.} \quad \underline{x} \in X$$

$$0.33 \left[\frac{540 - (24x_1 + 27x_2 + 0x_3 + 15x_4 + 1.1x_5 + 52x_6)}{540 - 93.34} \right] \leq d_\infty$$

$$0.33 \left[\frac{(10x_1 + 20x_2 + 120x_3) - 8.44}{110 - 8.44} \right] \leq d_\infty$$

$$0.33 \left[\frac{(.225x_1 + 2.2x_2 + .8x_3 + .1x_4 + .05x_5 + .26x_6) - 2.26}{6.29 - 2.26} \right] \leq d_\infty$$

This is a LP problem. The solution is:

$\underline{x}$ = (3.08, 0, 0, 10.00, 8.63, 4.00)

$\underline{f}$ = (441.43, 30.81, 3.15)

The solutions of the three cases are presented to the DM. The DM may pick any one of them, or may not agree with any of them.

Notes:

(1) The solutions of the nutrition problem are summarized in the following table. Also in the table is a set of solutions for a given set of weights [0.30, 0.20, 0.50] for $[f_1(\underline{x}), f_2(\underline{x}), f_3(\underline{x})]$. It is obvious that when the weight for $f_3(\underline{x})$, the cost objective, is 0.50 rather than 0.33, the cost is reduced to 2.32 from 2.65 for p=1; to 2.65 from 2.92 for p=2, and to 2.8 from 3.15 for $p=\infty$.

		f_1	f_2	f_3	x_1	x_2	x_3	x_4	x_5	x_6
Positive ideal solution		540.00*	8.44*	2.26*						
Negative ideal solution		93.34⁻	110.00⁻	6.29⁻						
	(0.33,0.33,0.33)	253.89	55.51	2.65	3.76	0.10	0.13	10.00	10.00	0
p=1	(0.30,0.20,0.50)	249.91	70.85	2.32	3.98	0.05	0.25	10.00	2.65	0
	(0.33,0.33,0.33)	420.93	27.75	2.92	2.48	0.13	0	9.94	1.08	4.00
p=2	(0.30,0.20,0.50)	417.70	29.15	2.65	2.48	0	0.03	10.00	0.60	3.99
	(0.33,0.33,0.33)	441.43	30.81	3.15	3.08	0	0	10.00	8.63	4.00
p=∞	(0.30,0.20,0.50)	440.48	34.37	2.80	3.44	0	0	10.00	0	4.00

(2) There can be infinitive solution sets for a MODM problem solved by the method of global criteria, because the global criteria p can be any integer between 1 and ∞. Among the infinitive p's, Yoon [1986] mentioned three meaningful p's, i.e.: p = 1, 2, and ∞. For p = 1, distance between two points is the longest distance in the geometric sense. For p = 2, distance between two points is the shortest distance in the geometric sense. For p = ∞, distance between two points is the shortest distance in the numerical sense. Three parameters, p = 1, 2, and ∞ are widely used to solve the MODM problem.

Since the solutions will be different among different p's, we'll face three different solutions. To choose a preferred solution depending on human cognitive power is not scientific nor reliable. Yoon [1986] introduces the relative reliability of a distance (d_p) measurement to reconcile several compromise solutions.

The relative reliability of d_p is written as:

$$\lambda_p = E(d_p)/\{E(d_1) + E(d_2) + E(d_\infty)\}$$

where each $\lambda_p = E(d_p)$'s, (mean of d_p), is obtained using the Monte Cario simuiation technique.

The three distance measurements (d_p) are combined by ietting λ_p be weights in a iinear combination form. The unified distance measurement (d) for each individuaI aiternative is caiculated by:

$$d = \lambda_1 d_1 + \lambda_2 d_2 + \lambda_\infty d_\infty$$

Each aiternative has a d vaiue. We compare the d's and choose the ieast one as our best solution.

(3) Advantages and disadvantages of this method.

The advantage of this method is that in the process of obtaining the soiution, the DM wiii not be disturbed by the anaiyst, which is preferred by the DM.

The disadvantage is that the anaiyst makes aii necessary assumptions about the DM's preference. This is a difficuit task even for weii trained and knowiedgeabie anaiysts.

(4) We wiii use a numericai exampie to iiiustrate that the soiution to equation (3.4) is equivaient to the soiution for $p = \infty$ for the Method of Giobai Criterion.

The PIS and NIS of the nutrition probiem, and an arbitrary set of nondominated soiutions is given as:

Numericai Exampie 2. (Nutrition Probiem)

	Max Carb	Min Chol	Min Cost
PIS	540.00	8.44	2.26
NIS	93.34	110.00	6.29
f	412.55	20.43	2.75

	Milk	Beef	Eggs	Bread	Lettuce	OJ
x = [	2.04	0.00	0.00	10.00	5.00	4.00]

If $w_1 = w_2 = w_2$ is assumed:

$$\min F_{\infty} = \left[\frac{540 - 412.55}{540 - 93.34} \right]^{\infty} + \left[\frac{20.43 - 8.44}{110.0 - 8.44} \right]^{\infty} + \left[\frac{2.75 - 2.26}{8.29 - 2.28} \right]^{\infty}$$

$$= [0.2853]^{\infty} + [0.1181]^{\infty} + [0.1216]^{\infty}$$

For $p = 50$ or $p = 100$ to replace $p = \infty$, we set:

$$F_{p=50} = 5.822 \times 10^{-28} + 4.026 \times 10^{-47} + 1.756 \times 10^{-46}$$

$$F_{p=100} = 3.390 \times 10^{-55} + 1.679 \times 10^{-93} + 3.114 \times 10^{-92}$$

For both $F_{p=50}$ and $F_{p=100}$, from the order of magnitude of point of view, the first term of the summation dominates, and the second and third terms can be neglected. Therefore, the $p = \infty$ problem is equivalent to minimization of the maximum dominated term, i.e., MINIMAX approach.

TOWARD RATIONAL DECISION MAKING USING THREE DIFFERENT MCDM METHODS

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ABSTRACT

Decision making problems which involve multiple, conflicting, and noncommensurable criteria are frequently encountered in both private and public organizations. Currently, decision problems are solved by managements using their intuitive judgements. Such a decision process is dangerous and inconsistent. To help the DM make reliable decisions, researchers have poured their efforts into forming various solution methods for all kinds of MCDM problems. The MCDM problems can be broadly categorized into (1) multi-attribute problems, and (2) multi-objective problems. Each category has different problem characteristics. None of the solution methods are capable of solving both kinds of problems.

In this study, this author discussed three solution methods, namely, TOPSIS-MADM, TOPSIS-MODM, and ISGP II. The first one deals with the multi-attribute problem. It utilizes a distance measurement function and attribute normalization function to obtain an preferred solution which is closest to the ideal solution and farthest to the negative ideal solution. The second one solves multi-objective problems. It is an extension of the method of Global Criteria and has properties inheritated from TOPSIS-MADM. The relative importance factor, $\underline{H}$, is introduced to combine two distance measurements: distance to positive ideal solution, and distance to negative ideal solution. The last one solves multi-objective problems too. It combines attractive features of the STEM method, the concept of Goal Programming, and an iterative approach to systematically derive a set of preferred

solutions. The DM is actively involved in the solution-finding process.

The formations of MCDM solution methods provide the DMs with systematic approaches to derive better decisions. Computerization of those solution methods enhances the power of the methods while reducing the burden of calculations, hence makes MCDM solution methods more realistic in real-world applications.